图论 2020 GTAC

北卡罗来纳州立

10~14 需重新学习

videos 在百度网盘

Set Operations

$A \cup B$ Union

$A \cap B$ Intersection

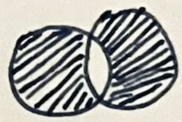
$A \setminus B$ Minus

if $A \subseteq X$, $\overline{A} = X \setminus A$ Complement

Symmetric Difference

$$A \oplus B = (A \setminus B) \cup (B \setminus A)$$

$$= (A \cup B) \setminus (A \cap B)$$



Cartesian Product

笛卡尔积，列出两集合元素的所有组合

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

Power Set

由集合 S 的所有子集构成的集合 Pow(S)

$$P(\{a, b\}) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$$

Sequenies

$$\{0, \cdots, n\} = [0 : n+1] = [: n+1] = [n+1]$$

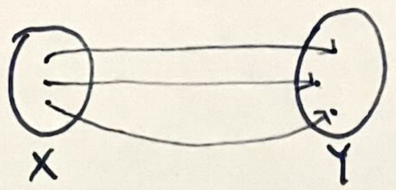
$\hookrightarrow$ n+1 个元素

Functions

$f: X \to Y$

从 domain X 至 co-domain Y 的投射方式

i.e. $f(x)$ 是 Y 中元素



Restricted f to $S \subseteq X$

$f|_S : S \to Y$

$f|_S(x) = f(x)$ for $\forall x \in S$

$f(S) = \mathrm{im} f|_S$

"限制定义域"

\circ

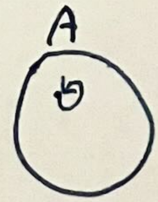
$g \circ f$

$: X \xrightarrow{f} Y \xrightarrow{g} Z$

$$(g \circ f)(\underbrace{x}_{\in X}) = \underbrace{g(\underbrace{f(x)}_{\in Y})}_{\in Z}$$

Identity Function

输入输出始终相等



$id_{\underline{A}}(x) = x$

定义域

即，$id_A : A \to A$

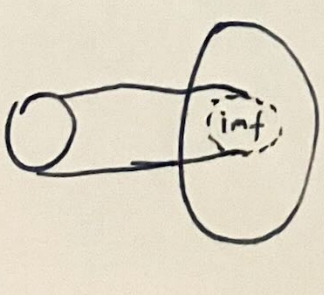
Image of f

$\mathrm{im} f = \{ f(x) \mid x \in X \}$

$\subseteq Y$

$f: X \to Y$

即，Y 中被 X 的投射覆盖的部分



$f: X \to Y$

Injection 单射

一对一，若 $x \neq x'$ 则 $f(x) \neq f(x')$

Surjection 满射

$\mathrm{im} f = Y$，每个 y 都可以找到至少一个 x 使 $f(x)=y$

Bijection 双射

即是单射又是满射

Bijection have inverses $f^{-1}: Y \to X$ 使 $f^{-1} \circ f = id$

Isomorphism 同构 $\cong$

Define $A \cong B$ iff $\exists\ A \underset{g}{\overset{f}{\rightleftarrows}} B$

s.t. $f \circ g = id_B$ and $g \circ f = id_A$

$\boxed{\text{Binary Relations}}$ 两个数学对象的联系, e.g. $2<3$

假设在集合 X 上有关系 R,也记为 $\sim$,以下写法一样:

$x, y \in X$

$$(x, y) \in R$$

$$x \sim y$$

$$([x, y], R)$$

一些备选性质(A中)

I I	Reflexive	$\forall x \in A,\ x \sim x$ "每个元素与自身关联"
	Symmetric	$x \sim y \Rightarrow y \sim x$
I I	Antisymmetric	$(x \sim y \text{ and } y \sim x) \Rightarrow x = y$
I I	Transitivity	$(x \sim y \text{ and } y \sim z) \Rightarrow x \sim z$
I	Complete "connex"	A中元素两两关联, all pairs connected in some way $x \sim y$ or $y \sim x$, for $\forall x, y \in A$

Partial Orders 偏序

e.g. $(\{a, b, c\}, \leq)$ 即 $a \leq b \leq c$

Total orders 全序,即特殊的偏序

上例也是一个全序,因所有set中元素都被包含

$\boxed{\text{Graph}}$ 由 顶点集 $V = \{a, b, c\}$ 与
边集 $E = \{\text{“}ab\text{”}, \text{“}bc\text{”}\}$ 组成

if “ab” $\in E$，则 a 与 b $\boxed{\text{adjacent}}$ (由edge相连)

if $a \in$ “ab”，则 a 与 边“ab” $\boxed{\text{incident}}$

$\boxed{\text{Degree}}$ of a vertex v

$$\deg(v) = \#\ \text{edges incident to } v$$
$$= \#\ \text{vertices adjacent to } v$$

Clique ($\boxed{\text{Complete Graph}}$ 的写法)

K_n = n个顶点的全连接子图

K_1

K_2 

K_3 

K_4 

一些特殊的Graph

Path P_n with n edges, n+1 vertex

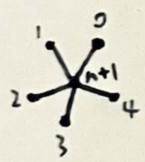
($V_i \to V_{i+1}$)

Cycle C_n with n edges, n vertex

($V_i \to V_{(i+1)\%n}$)

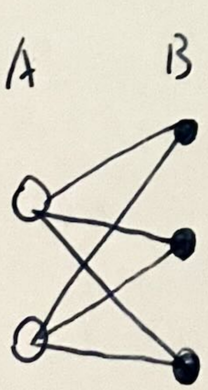
Star S_n with n edges, n+1 vertex

($V_i \to V_n$)



Bipartite Graph $K_{a,b}$ with a vertex in A

b vertex in B



全连接/Complete Bipartite Graph

有 $a \times b$ edges.

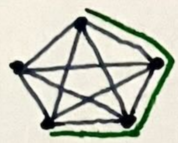
($V_i \to V_{a+j}$) $i \in [a], j \in [b]$

G is a $\boxed{\text{Subgraph}}$ of H iff

$V_G \subseteq V_H$

and $E_G \subseteq E_H$ ，记为 $G \subseteq H$

e.g. $P_3 \subseteq K_5$



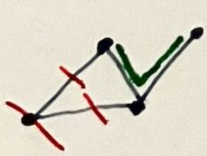
e.g. $S_3 \nsubseteq P_3$ 因 $E_{S_3} \nsubseteq P_{S_3}$

即 $E_{S_3} \setminus E_{P_3} \neq \emptyset$

如何得到 Subgraph?

Remove Edges $\longrightarrow$ Spanning subgraphs

Remove Verticies & incident edges $\longrightarrow$ Induced Subgraphs



假设有一组 $G \to H$ 的映射 $f = (f_V, f_E)$

$$f_V : V_G \to V_H$$

$$f_E : E_G \to E_H$$

本质上是 Vertices 上的 function

即 $f_E(\text{"}uv\text{"}) = \text{"}\underline{f_V}(u)\ \underline{f_V}(v)\text{"}$

$G \to H$ 双射, G is <u>Isomorphic</u> to H $\quad G \cong H$

$G \to H$ 单射, G is <u>Homomorphic</u> to H

Graph Isomorphism 的定义与 set 很像, recall: $g \circ f = id_G$

只是需同时满足 V 与 E.

$$G \underset{g}{\overset{f}{\rightleftarrows}} H$$

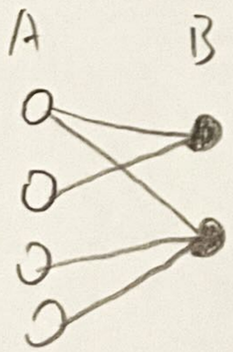
$$id_G = (id_{V_G}, id_{E_G})$$

$$g \circ f = (f_V \circ g_V, f_E \circ g_E)$$

例] Homo

G is bipartite iff $\exists f : G \to K_2$

G



$$f_V(u) = \begin{cases} 0 & \text{if } u \in A \\ 1 & \text{if } u \in B \end{cases}$$

K_2

"0 $\quad$ 1" in K_2 & "0 $\quad$ 0" not in K_2

例 Homo

Assign colors to V_G，令$\boxed{\text{相邻顶点颜色不同}}$

$$C_V : V_G \longrightarrow \{C_1, C_2, C_3, \dots C_p\}$$

$$\text{s.t. } \boxed{``uw" \in E_G \Rightarrow C_V(u) \neq C_V(w)}$$

G $\qquad$ K_3

于是 p-color (color set 大小为 p) 可以用来 check 两图是否 Homo.

!!! $C = (C_V, C_E)$ is a homomorphism 映射

!!! 定义 $\boxed{\chi(G)}$ 为 p-color G 所需至少的颜色数

Iso 意义下. $f: G \xrightarrow{\cong} H$

有 $c: H \to K_p$ 为 a p-coloring of H

则 $c \circ f: G \to K_p$ 为 a p-coloring of G

反方向也可推. 总之, Iso 意义时, $\chi(G)$ 不变

$\boxed{\text{Graph Invariant}}$ 指 Isomorphism 变化后不改变的参数.

只能用来反证两图不 Iso

(e.g. ? —> 不 Homo —> 不 Iso)

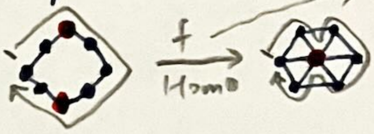
\# vertices

\# edges

Degree 列表

Bipartite or not ($\chi(G)$ of p-coloring)

有无特定子图: Triangle (K_3) , cycle (Injective walk



是否连通

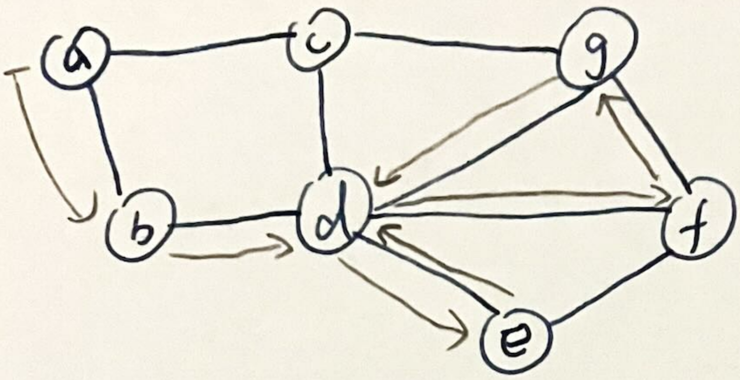
Functional:

$$G \cong H \implies geom(G) \cong geom(H)$$

Isomorphism of graph　　　　Homeomorphism of geom realization

A $\boxed{\text{walk}}$ in G of lengh k,

是一串序列 " $v_0 v_1 \dots v_k$ " 记录经过的顶点



(a,d)-walk，由 a 始至 d 结束

e.g. abdedfgd

连通图：两两可达

G is $\boxed{\text{connected}}$ if for all $u,v \in V_G$,

There is a (u,v)-walk in G

A $\boxed{\text{Path}}$ in G is an injective walk,

即不重复任何顶点的 walk.

Euler walk

其它walk - Trail ：顶点可复用，但边不可重复

- Tour ：回到原点的 Trail closed Euler walk

i.e. 一种 closed walk ，$w_V(0) = w_V(k)$

- Cycle ：closed Path

"robustness of connectivity"

k-connective 不存在size ≥ k-1 的 S，使 G\S 不连通 !!!

一些 G\S 后仍保持连通，

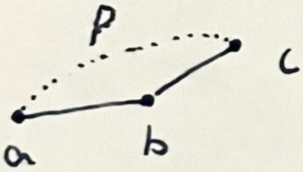
移除的顶点集 S 不能大于 k-1 (即少于K个)

此时 $\min_{v\in V_G} \text{degree}(v)$ 一定 ≥ k

2-connected 时，every edge is part of a cycle

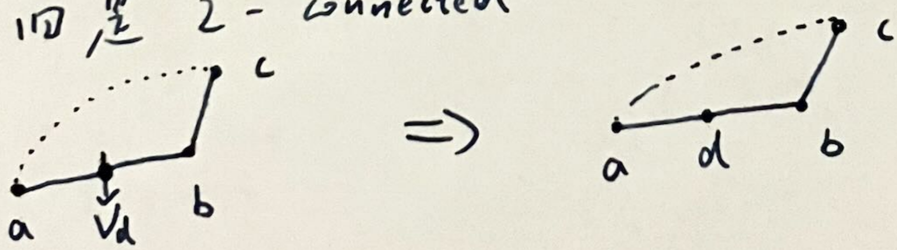
证明：由于去除任一个v (G\b) 后依旧 connected,

说明 a c 间还有已知/未知的 path.



如果对 2-connected 图中 split an edge,

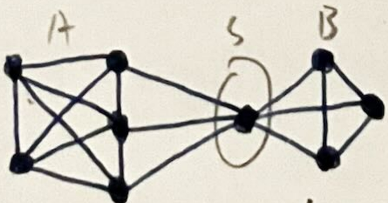
之后仍是 2-connected



Not 2-connected 例

这个例子中，(不动 cut vertex 先!)

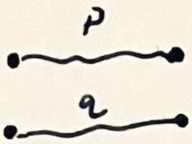
K=1



cut vertex / (A,B)-separator

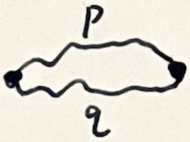
size ≥ K

disjoint path



无交集 $\text{im}\, p \cap \text{im}\, q = \phi$

independent path



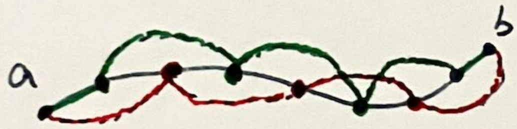
仅首尾相合 $\text{im}\, p_V \cap \text{im}\, q_V = \{a, b\}$

$\text{im}\, p_E \cap \text{im}\, q_E = \phi$

Menger's Theorem if G is k-connected, 每一对顶点间都存在 k 条 independent path (至少?)

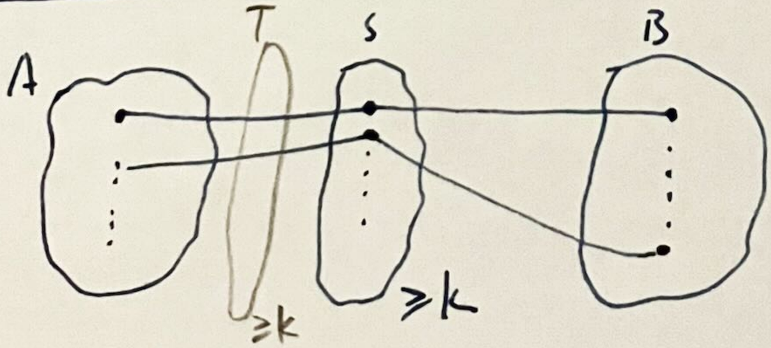
也可推导 G is k-conn

以 2-conn 为例,



若 $G \setminus S$ has no path between (A, B), 则 set S 是

(A, B)-separator



minimum AB-seperator of size(k) $\Rightarrow$ exist (k) disjoint AB path

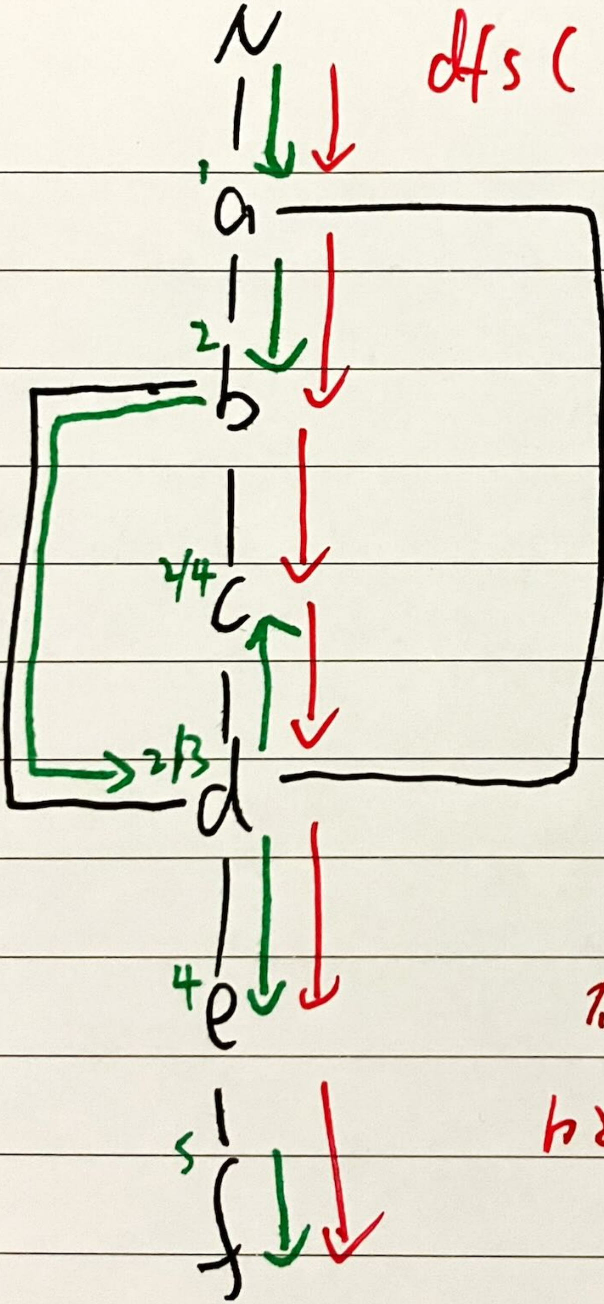
e.g. "a, b_2"

"a, b_1"

...

令 T 为 (AS-seperator)$^{\min}$, 它也分离 AB,

故 $|T| \geq k$



dfs(G, a) 运行完

似乎与定义G的顺序有关??

从 ToVisit[] 中取下一个V

Neighbor 加入 ToVisit[]

在同一条dfs找到的path中?

V_1 与 V_2 connected!

A $\boxed{\text{Forest}}$ is a graph with no cycles

i.e. $\forall a,b \in V$ $\exists$ $\boxed{\text{unique}}$ (a,b)-path

A $\boxed{\text{Tree}}$ is a connected Forest

A $\boxed{\text{leaf}}$ is a vertex of degree 1 (in a tree)

Every non-leaf vertex in a tree is a cut vertex

否则，若 $T-v$ connected，说明有 cycle (once Path 存在)

A tree plus an edge : form a cycle , not a tree now.

$T \subseteq G$ is a $\boxed{\text{spanning tree}}$ of G, if

- T is a Tree
- T is spanning ($V_T = V_G$) 覆盖 G 中所有 Vertex

每个连通图都会有一个生成树

如何得到？不停从 cycle 中 remove edge.

反之，制造 unique circles.

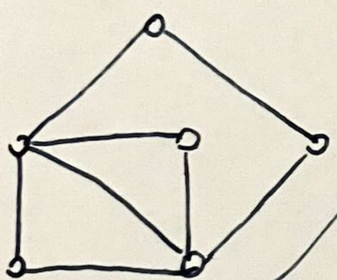
七桥问题：走遍7座桥，每座恰好经过一次？

不行！

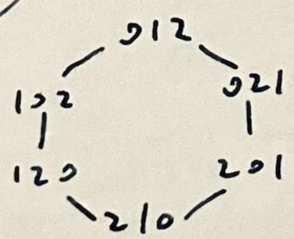
问题 $\Longrightarrow$ 可否形成 closed Euler walk
（回原点，边不重用）

sufficient 条件

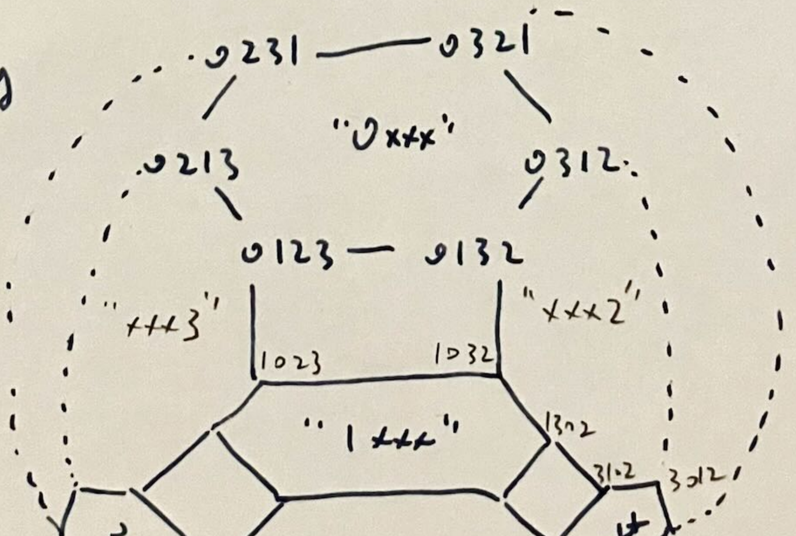
- connected graph ✓
- all degrees even ⑦✗



3! 个 Vertex 时



4! 个 Vertex 时



设E是度量空间X的一个子集，如果P的某个邻域包含在E中，则点P是E的$\boxed{\text{内点}}$

如果度量空间的一个子集的所有点都是该集合的内点，则这个子集是一个$\boxed{\text{开集}}$

$\forall p\in E$，$\exists r>0$ 使得 $N_r(P)\subset E$ $\Longrightarrow$ $E\subset X$ 是开集

设E是度量空间X的一个子集，如果P的~~某个~~每个邻域都包含E的至少一点（P本身除外），则P是E的$\boxed{\text{极限点}}$。此时对于任意$r>0$，邻域$N_r(p)$包含E中无穷多个点。

（所以，有限集没有极限点）闭集

如果度量空间的一个子集包含其所有极限点，则该子集是一个$\boxed{\text{闭集}}$

$\{p\in X \mid p\text{是}E\text{的极限点}\}\subset E$

$\Longrightarrow$ $E\subset X$ 是一个闭集

广义定义

假设X是一个集合，如果存在一系列X的子集满足以下条件，则每个这样的子集称为X的一个开集，X称为拓扑空间

- $\emptyset$ 和 X 为开集
- 有限多个开集之交为开集（无穷多个开集的交集未必开）
- 任意多个开集之并为开集

特殊开集：拓扑

Let X be a set, A $\boxed{\text{topology } T}$ on X is a collection of subsets of X. each called an open set

（更广义的开集定义）

Topological spaces C 是拓扑 T 的集合，即上文 open sets 的集合.

可以有不同的拓扑空间.

如何判断 C 是否是 X 上的拓扑？（用拓扑定义）

e.g. $X = \{a, b, c\}$

$C_1 = \{\phi, \{a\}, \{a, b\}, X\}$ 是开集，是 X 上拓扑）

$C_2 = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, X\}$ 中，

$\{a\} \cup \{c\} = \{a, c\} \notin C_2$

不是开集，也因此不算 X 上拓扑

在 R^n 空间中定义开集，即

$X \subseteq R^n$ is open if it is a union or finite intersection of open balls

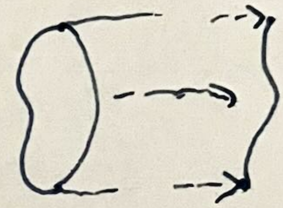
$$\{x : \|x-c\| < r\}$$

~~若 $f: X \to Y$ 的原像集~~

(对于Y中的任意开集)

$f: X \to Y$ is $\boxed{\text{continous}}$ if for $\forall$ open sets $S \subseteq Y$,
$f^{-1}(S)$ is open in X

(其原像在X中是开集)



- id_X is continous
- If f and g are continous, $f \circ g$ is continous

一些连续函数：

Constant Functions

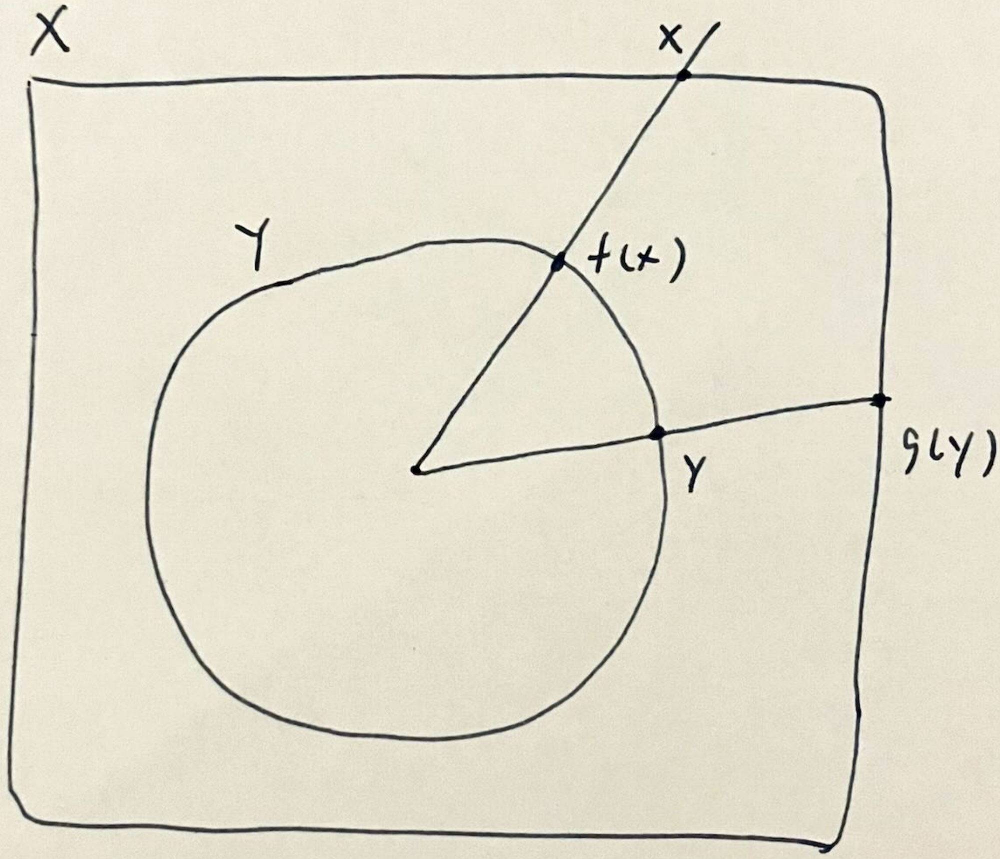
Linear Functions

Homeomorphism $\cong$ between geom(G)

(Isomorphism for topological spaces)

$$X \cong Y \quad \text{iff} \quad X \underset{g}{\overset{f}{\rightleftarrows}} Y$$

$$\text{s.t. } g \circ f = id_X \qquad f \circ g = id_Y$$



x , y

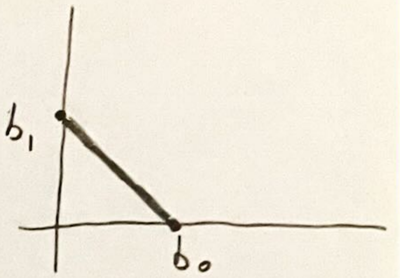
" geometrically different ,

Topologically equivalent "

$G=(V,E)$ 的 Geometric Realization 中，以点表示 V，以线段表示 E.

Graph $\to$ Topology
(Iso) (Homeo)

$\underline{\text{geom}(G)}$: $\underbrace{\left(\bigcup_{v\in V} v_i\right)}_{\text{points}} \cup \underbrace{\left(\bigcup_{E} \overline{v_iv_j}\right)}_{\text{line segments}}$



以 n 个 vertex 的 G 为例，R^2 空间中

$v_i = \begin{bmatrix}0\\ \vdots\\ \vdots\\ 1\\ 0\end{bmatrix}_i$ edge $\overline{v_iv_j} = \{(1-t)v_i + tb_j\} \mid t\in[0,1]\}$

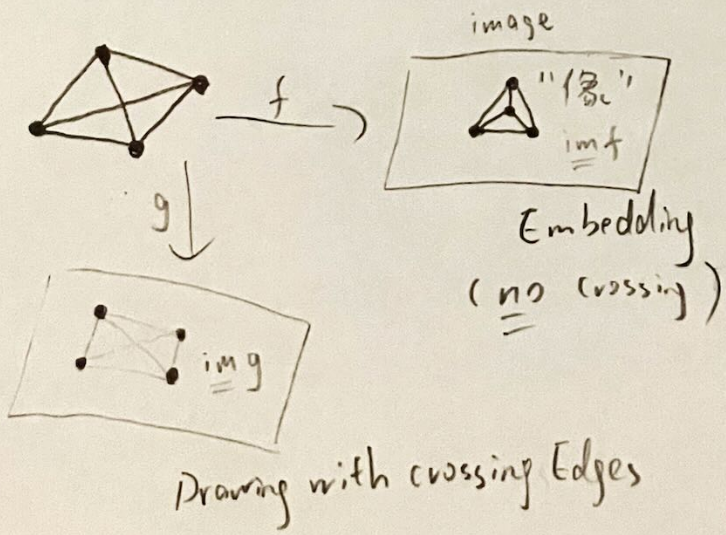
(linear map , continuous)

$\boxed{\text{Drawings}}$ will be continous maps from geom(G)

$\text{geom}(G) \to R^d$

$\boxed{\text{Embeddings}}$ will be injective drawings (No crossings)

以 $\text{Geom}(K_4) \to R^2$ 为例



A graph G is $\boxed{\text{planar}}$ if

$\exists$ embedding $\text{geom}(G) \to R^2$

($d=2$ 定义：平面)

反之，K_5，$K_{3,3}$ 会作为子图出现在每个非平面图中

关于 "Every graph can be embedded in $\mathbb{R}^3$"

可在 $\mathbb{R}^3$ 中任取 4 个不在同一平面上的点

此时 $\overline{ab}$ 与 $\overline{cd}$ 无 crossing

但若有 crossing，只能说明它们在同一平面上

If $G \cong H$ then $geom(G) \cong geom(H)$

证明：$G \underset{g}{\overset{f}{\rightleftarrows}} H$ s.t. $g \circ f = id_G$ $\quad f \circ g = id_H$

$$\Downarrow$$

$$geom(G) \underset{geom(g)}{\overset{geom(f)}{\rightleftarrows}} geom(H)$$

$$\begin{aligned} geom(g) \circ geom(f) &= geom(g \circ f) \\ &= geom(id_G) \\ &= id_{geom(G)} \end{aligned}$$

同理 $geom(f) \circ geom(g) = id_{geom(H)}$

于是~~进~~得 $geom(G) \cong geom(H)$

($\because$ 双射)

If f is an embedding of G, then

$$\underline{\mathrm{im}\, f} = f(\mathrm{geom}(G)) \ \underline{\cong \ \mathrm{geom}(G)}$$

证.

指仅取 image 为 co-domain，所以满射

let f' be the corestriction of f to $\mathrm{im}\, f$，即

$$f' : \mathrm{geom}(G) \to \mathrm{im}\, f$$

by its definition，f' 单射、满射、连续

也已知，R^n 中 f'^{-1} 也连续

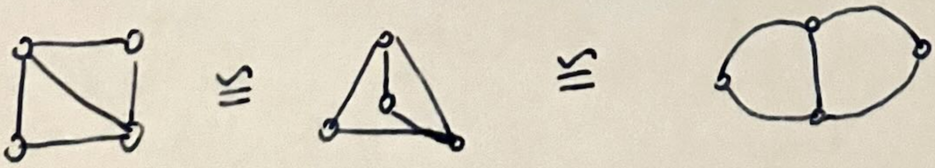
于是 f' 是 homeomorph

(基于 Embedding 的定义)

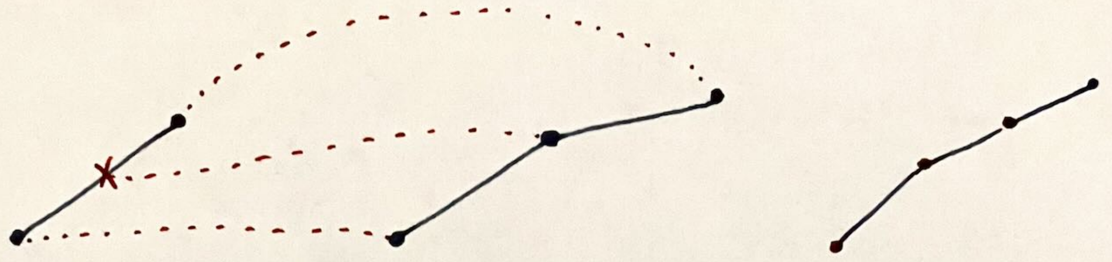
推论：

Any two embeddings of the same graph will be homeomorphic

$$G \longrightarrow \mathrm{geom}(G) \underset{f'_2}{\overset{f'_1}{\rightrightarrows}} R^2$$



当 $G \neq H$ 时，也可以有 $geom(G) \cong geom(H)$



$$G \xrightarrow{f} H \xrightarrow{\text{included in}} Z$$

f: subdividing 0 or more edges of G,

此处 H 是 G 的一个 subdivision.

若 H 是 Z 的一部分 (i.e. Z contains a subdivision of G)

简约!!

则 G is a topological minor of Z

例一 从 G 开始逐步切 Edges，可归纳：If $geom(G) \cong geom(H)$, there exists a graph K that is a subdivision of both G and H

例二 由于 C_3 含 cycle (非树)，于是其 subdivision 也非树，即：G is a forest iff it contains no C_3 topological minor

Linear Embeddings

$$f: \underset{\subseteq R^n}{geom(G)} \longrightarrow R^d$$ 若是线性投射，

则 f 会是一个 matrix

$$M^{d\times n} = \underbrace{\begin{bmatrix} | & | & | & | & | & | \\ | & | & | & | & | & | \\ | & | & | & | & | & | \end{bmatrix}}_{n \text{ cols} = \text{positions for the } n \text{ vertices}} \quad \text{so} \quad Mv_i = p_i$$

（也即 V 在 R^d 中位置）

" $f = M|_{geom(G)}$ is a linear Drawing "

（若 Injective，则 Embedding）

i.e. $\text{Im} f$ 中无 crossing E

A **polygon** is the image of a linear embedding of a cycle $\to R^2$

A **polygon path** is the image of a linear embedding of a path $\to R^2$

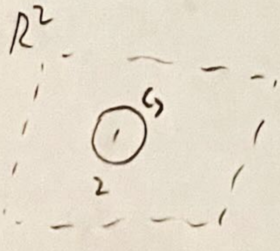
an equivalence relation

$a, b \in X$ are **path-connected** if there exists a polygonal path into X that starts at a and ends at b

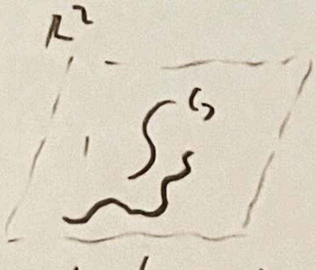
相互间 path-connected 的元素所组成的合集

称 **path-connected components** an equivalence class

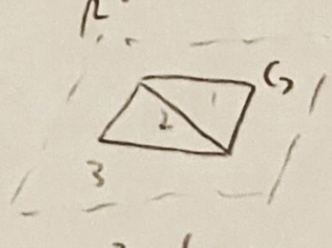
e.g. 对于 R^2 中的 Embedding G，被 $R^2 \setminus G$ 分割得到的 path-connected components 被称为 **faces**



2 faces



1 face, 2 connected components



3 faces

cycle $\to R^2$ 即 **Jordan Curve**，跨越faces间则必与之相交

在被 Jordan Curve 切割的空间中，如何判断2个点 x、y 是否 path-connected？

仅适用于 Jordan Curve！

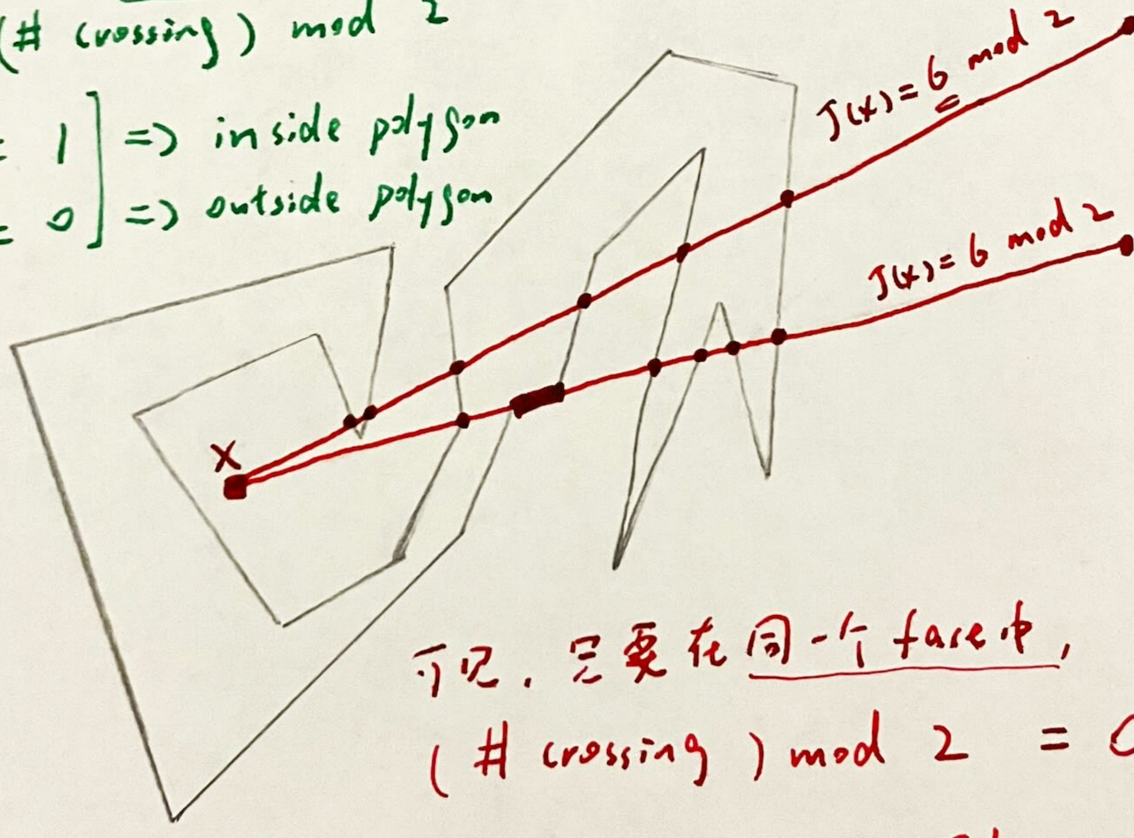
- cycle $\rightarrow \mathbb{R}^2$ 称 polygon，不适用 e.g. 

- plane 中，不适用 e.g. 莫比乌斯环

1. 从 x 发射一条$\boxed{\text{射线}}$，射线方向不重要

   $J(x) = (\#\ \text{crossing}) \bmod 2$

2. $\left.\begin{array}{l} J(x) = 1 \\ J(x) = 0 \end{array}\right]$ $\begin{array}{l} \Rightarrow \text{inside polygon} \\ \Rightarrow \text{outside polygon} \end{array}$



可见，只要在同一个 face 中，

$(\#\ \text{crossing}) \bmod 2 = 0$

即 Even number of Edge crossing

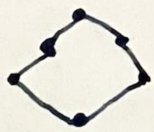
3. $J(x) = J(y)$ iff x, y are path connected

A graph is $\boxed{\text{outerplanar}}$ if there exists an embedding into the plane that places all vertices on the outer face

i.e. 延伸至无穷远处的那个区域.

Outerplannar
(无内部点)

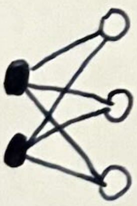
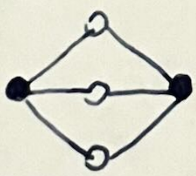




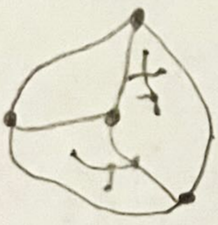
G　　Sub

若 G 的 subdivision 是 Outer-planar，
则 G 也一定是 Outer-planar （因 geom(Sub) ≅ geom(G))

Not Outerplannar : Edges shared by 2 triangles
(inner faces)


i.e.

$K_{2,3}$

!!! planar 中
!!! R^2



K_4

G is outerplannar iff it contains no

K_4 or $K_{2,3}$ subdivisions

i.e. not a topological minor of G's subgraphs

$\boxed{\text{Topological Invariant}}$ 在 Homeomorphism 变化后不改变

i.e. $geom(G) \cong geom(H) \Longrightarrow e(G) = e(H)$

例如，$e(G) = |V| - |E|$ 在 subdivision 后值不变

$e(\text{Cycle}) = 0$ ，$e(\text{Tree}) = 1$

当 G is $\overset{(R^2\text{中的 Image 无 crossing})}{\text{planar}}$ with $\underline{\underline{k}}$ connected components,

then for all embeddings of G,

$$\overset{n}{|V|} - \overset{m}{|E|} + |\text{Faces}| = 1 + k$$

即是 " $\boxed{\text{Euler's Formular}}$ in the $\boxed{\text{plane}}$ "

直觉：加 Edge 时，会发生 Face +1 或 k −1

若加入任意边e后，G+e都会变为Not Planar，

则称G是一个 $\boxed{\text{maximal planar graph}}$.

它的每一种embedding都是一种 $\boxed{\text{triangulation}}$.

A triangulation is an embedding of a planar graph in which every face is bounded by 3 edges.

faces都呈三角形

$\boxed{\text{Maximal 时}}$

$$|E| = |V| + |Faces| - 2$$ (Euler's Formular)

$$\boxed{\text{解得}\quad |E| = 3|V| - 6}$$

其中 $|Faces| = \frac{|E|}{3} \cdot 2$ 代入原式

满的 Triangles (三边) share 2条边 ?

总之，小于Maximal，G+e才可能仍planar.

Every maximal planar graph with at least 4 vertices is 3-connected

证明：

已知，它的每一个面都会是三角形.

则<u>移去任一顶点</u>，<u>新生成的face</u>会是a <u>polygon</u>

即 $G \setminus v$ is <u>2-connected</u>

则，G is 3-connected

Let G be an embeded planar graph

$$G = im(\, geom(G) \longrightarrow R^2 \,)$$

If G is 2-connected then every face of G is bounded by a cycle

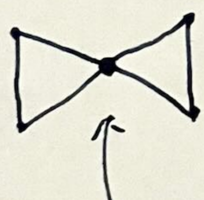
证明：

假如有 face not bounded by a cycle，

即，沿边 walk 时，会有顶点被路过 2 次（定义）

Impossible for 2-connected

类似：



这个顶点是 cut vertex

Induced cycle 中，没有连接非相邻顶点的边

i.e. "No chords"

induced

not induced

A subgraph $S \subseteq G$ is non-separating if $G \setminus S$ is connected

即移除 S 后，G 依旧连通.

Let G be 3-connected ① and planar

Let G be an embedding of G.

A cycle C bounds a face of G iff

C is a non-separating ③ induced cycle ②

Faces in Graph

$\Downarrow$

Faces in Embedding

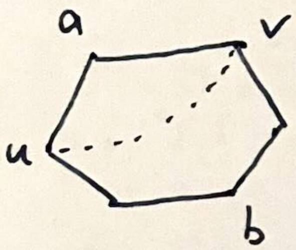
① 反例：if G is not 3-connected.

If C bounds a face, then C is an induced cycle

证②.

若 C is not induced, 有一条 uv-chord.

ab-paths 无论如何都会与之相交



于是 G \ {u, v} not connected

即, G not 3-connected

If C bounds a face, then C is non-separating

证③ Let u, v be vertices not in C

对于 3 (k) - connected graphs,

Menger's Theorem 认为

有 3 (k) 条 independent uv-paths.

而显然, 经过 cycle 的只有2条.

于是, 去除 cycle 后, G \ C 依然连通.

证④ if c is non-separating and induced, then c bounds a face

$\backslash c$ 与 connected　　polygon

四步证毕.

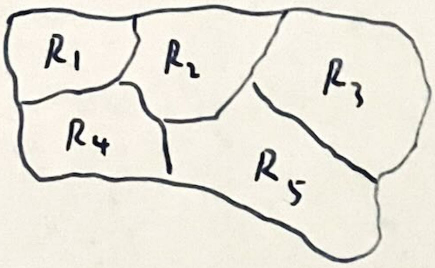
这个定理告知，对于 3-connected planar graphs，即使未进行 embedding（对于任何 embedding），也可知 which cycle will bound faces.

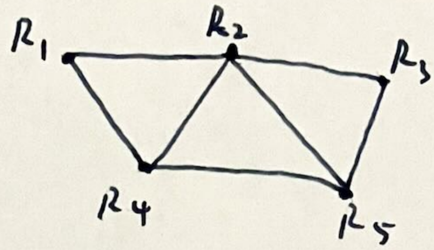
假设有一副<u>平面图</u>(e.g. Map)，可以将它转换成 $\boxed{\text{Dual Graph}}$

只 share 一条!!!

G中 <u>Reagons (faces) 为顶点</u>，若G中有 <u>shared edges</u> 则 G^* 中也相连.

cycles that bound faces





$$G \longrightarrow G^*$$

$$G^{**} \longleftarrow$$

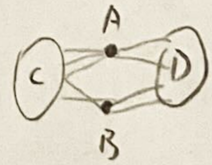
需证明: If G is a planar, 3-connected graph, then G^* is also <u>planar</u> and <u>3-connected</u>

找一个H，同时包含:

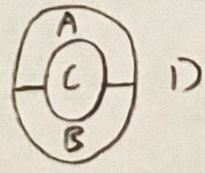
subdivision of G 与

subdivision of G^*

if not，假设 G^* 中:



则原图G只能:



A、B为了包住C，必须 share 2-edges，但如此就不是 3-connected

To prove: $G^{**} \cong G$

A <u>drawing</u> of G

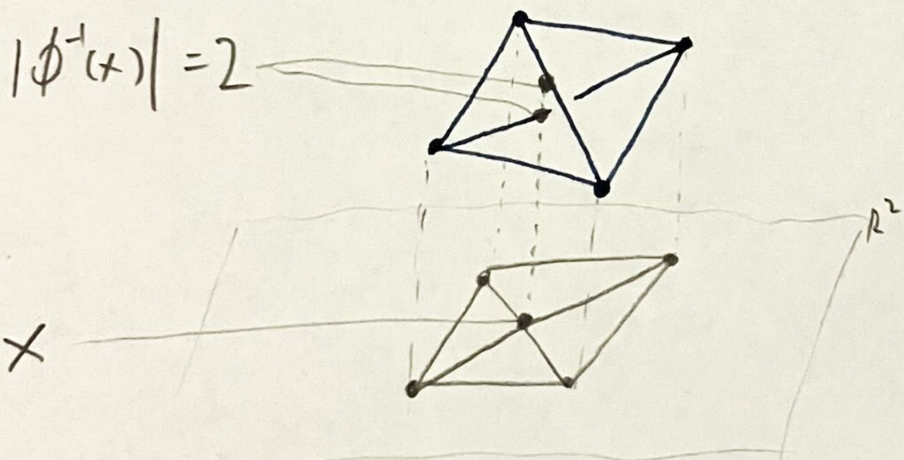
$$\phi : \mathrm{geom}(G) \to \mathbb{R}^2$$

is $\boxed{\text{non-degenerate}}$ iff

$\forall x \in \mathbb{R}^2, |\phi^{-1}(x)| \le 2$ 一处只容纳一组 cross

and

$\forall x \in \mathrm{im}\,\phi_V, |\phi^{-1}(x)| = 1$ <u>vertices</u> 不堆叠 (不 cross)



定义 The $\boxed{\text{crossings}}$ in a non-degenerate drawing ϕ are the points $x \in \mathbb{R}^2$ s.t. $|\phi^{-1}(x)| = 2$

定义 The <u>crossing number</u> of G $\boxed{cr(G)}$ is the minimum number of crossings of any non-degenerate drawing of G

If G is planar then $cr(G) = 0$

"embedding"

If $G \subseteq H$, then $cr(G) \leq cr(H)$

证明

let $f: geom(H) \to \mathbb{R}^2$ be the non-degenerate drawing that minimizes crossings.

$$\begin{tikzcd} geom(G) \arrow[r, "g"] \arrow[rr, bend right, dashed, "f \circ g"'] & geom(H) \arrow[r, "f"] & \mathbb{R}^2 \end{tikzcd}$$

$$\left|(f \circ g)^{-1}(x)\right| = \left|g^{-1}(f^{-1}(x))\right| \leq f^{-1}(x) \qquad \frac{\because g \text{ 是单射}}{\therefore \leq \ \subseteq}$$

$$im(f \circ g)_v = f(im\, g_v) \leq im\, f_v$$

由于 $f \circ g$ 只能得到 f 的子集，所以它也是 non-degenerate

G, 且.

$cr(G) \leq cr(H)$

证明 Crossing number is a graph invariant

Suppose $f : G \xrightarrow{\cong} H$

由于 f 满足单射 $G \to H$，于是 $cr(G) \le cr(H)$

也满足反向的单射 $H \to G$，于是 $cr(H) \le cr(G)$

综上，$cr(G) = cr(H)$

证明 Crossing number 是一个 Topological invariant

i.e. $geom(G) \cong geom(H)$

$$\Downarrow$$

$$cr(G) = cr(H)$$

关于 crossing number 的下限.

$\boxed{\text{Trival}}$ 假设每一对 edge 都不处于同一直线，且至多相交一次.

$$cr(G) \le \binom{|E|}{2}$$

$$\le \binom{\binom{|V|}{2}}{2} \le \binom{n^2}{2} \le n^4$$

$\boxed{\text{Euler's}}$ Formular 对于 maximal planar graph,

已知得 $|E| = \underline{3|V| - 6}$

多出这个数的 edges 与其它 edge 相交（至少一次）

即. $cr(G) \ge |E| - 3|V| + 6$

对于 sparse graph，这个 bound 更 tight，例如：

$$|E| > 4|V| \implies cr(G) > |V| + 6$$

$$\text{Full} \longrightarrow |E| \approx |V|^2 \implies cr(G) \ge \Omega(|V|^2)$$

$$H \subseteq G \longrightarrow |E| \approx |V|^p \implies cr(G) \ge \frac{1}{p^4} Cr(H)$$

keep each vertex with probablity $P = \frac{4|V_G|}{|E_G|}$

$$\ge \frac{1}{p^4}\left(|E_H| - 3|V_H|\right)$$

$$\ge \frac{1}{p^4}\left(p^2|E_G| - 3p|V_G|\right)$$

$$\ge \frac{|E_G|^3}{64|V_G|^2}$$

Simplex, $\boxed{\text{单纯形}}$ 指由 $(k+1)$ 个顶点构成的 k 维凸多面体 e.g. 线段 (dim=1), 三角 (dim=2), 四面体 (dim=3) ...

Simplicial Complex, $\boxed{\text{单纯复形}}$ 由有限个 Simplex 组成，其中任意两个单纯形交集为空或公共面，且每个单纯形的所有面属于该复形 (i.e. 每个面也是复形中单纯形
i.e. 凸
i.e. closed)

以 Graph = (V, E) 为例，

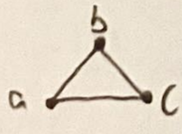
它的 Simplical Complex (V, S)

vertices　　simplices

$$S \subseteq Pow(V)$$

S is closed under:

$$T \subseteq \tilde{T} \in S \Rightarrow T \in S$$



G

$$simp(G) = (V_G, \{\emptyset\} \cup \{\{u\} \mid u \in V_G\} \cup E_G$$

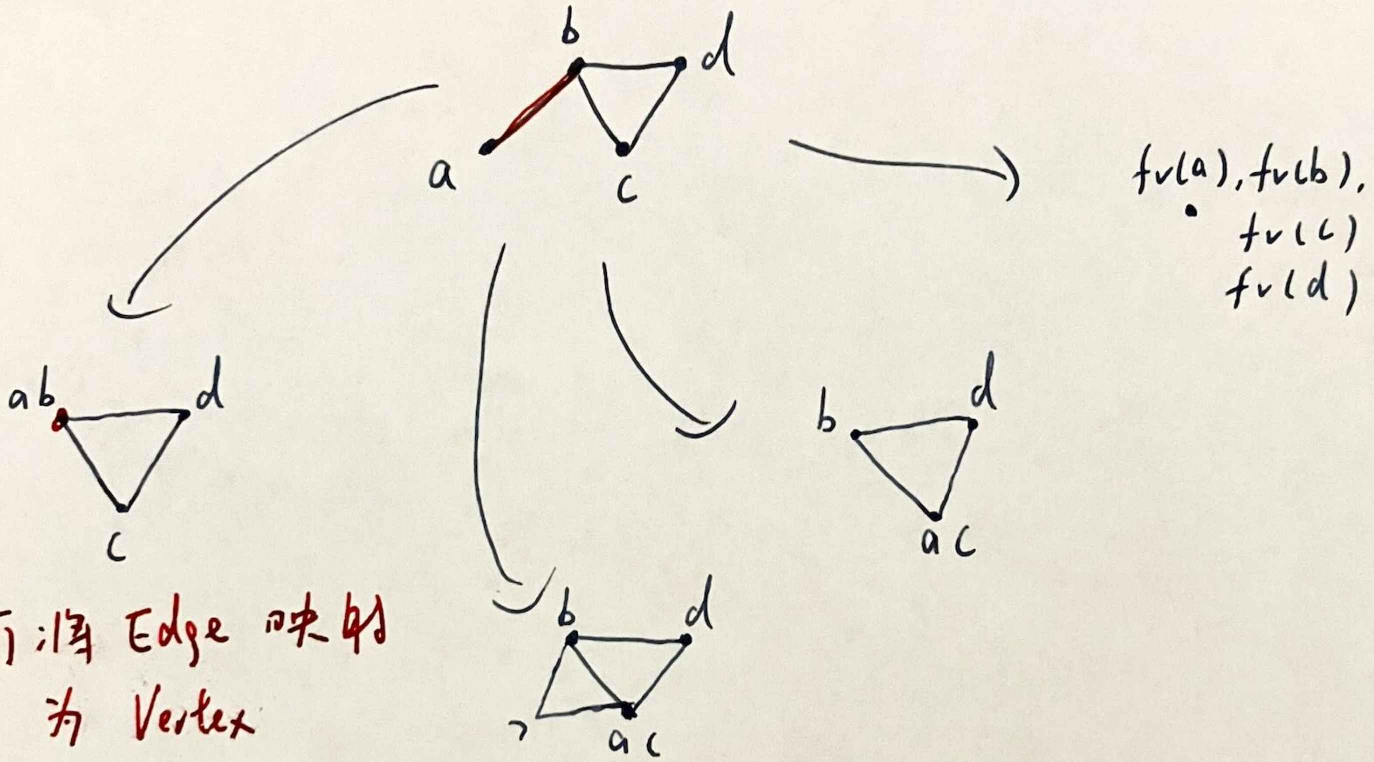
empty set

vertices

edges

定义 A simplical map $f: K \to L$ 为一组关于V、S的方程

$$\begin{cases} f_V: V_K \to V_L \\ f_S: S_K \to S_L \end{cases}$$ where $f_S(\underbrace{\sigma}_{\text{原图顶点集}}) = \{f_V(u) \mid u \in \sigma\}$



可将 Edge 映射为 Vertex

= 不一定会是满射

Simplical map 满足:

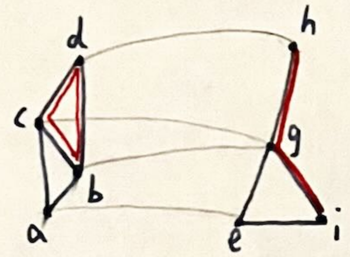
identity $id_K = (id_{V_K}, id_{S_K})$

composition $K \xrightarrow{f} L \xrightarrow{g} M$

则 $g \circ f = (g_v \circ f_v, g_s \circ f_s)$

$$(g_s \circ f_s)(\sigma) = g_s(\{f_v(u) \mid u \in \sigma\})$$
$$= \{g_v(f_v(u)) \mid u \in \sigma\}$$

1.1 Simplical Map $f: K \to L$ 为示例



$K \xrightarrow{f} L$

$f_V : V_K \to V_L$

$a \mapsto e$
$b \mapsto g$
$c \mapsto g$
$d \mapsto h$

$f_S : S_K \to S_L$

$bcd \mapsto gh$
$bc \mapsto g$
$cd \mapsto gh$
$ac \mapsto eg$
$ab \mapsto eg$

$\mathrm{im} f := (\mathrm{im} f_V, \mathrm{im} f_S)$

$\mathrm{im} f_V = \{e, g, h\} \subseteq V_L$

$\mathrm{im} f_S = \{\phi, e, g, h, eg, gh\} \subseteq S_L$

$\boxed{\text{“ im} f \text{ is a subcomplex of } L\text{”}}$

& $\dim(\mathrm{im} f) \le \dim(K)$

取 L 的 subcomplex $X \subseteq L$，其 K 中原像 $f^{-1}(X) = [f_V^{-1}(V_X), f_S^{-1}(S_X)]$

例.1 ① $X = (\{g\}, \{\phi, g\})$

$f^{-1}(X) = [\{b, c\}, \{\phi, b, c, bc\}]$

例.2 ② $X = (\{g, h, i\}, \{\phi, g, h, i, gh, gi\})$

X·图 $f^{-1}(X) = [\{b, c, d\}, \{\phi, b, c, bc, d, cd, bd, bcd\}]$

$\boxed{\text{∴ “} f^{-1}(X) \text{” is a subcomplex of } K}$

定义 Simplical complexes K and L are isomorphic ($K \cong L$) if there exist

$$K \underset{g}{\overset{f}{\rightleftarrows}} L \quad \text{s.t.}$$

$$g \circ f = id_K \ , \quad f \circ g = id_L$$

事实 Every graph homomorphism $f : G \to H$ induces a simplical map

$$simp(f) : simp(G) \to simp(H)$$

即, $simp(f) = (f_V, f_S)$ where

$$f_S(\sigma) = \{ f_V(u) \mid u \in \sigma \}$$

也易知

$$simp(id_G) = id_{simp}(G) \ ,$$

$$simp(f \circ g) = simp(f) \circ simp(g)$$

The <u>clique complex</u> of graph G is the <u>simplical complex</u> $\mathrm{clique}(G) = (V_G,\ \underline{\text{cliques in } G})$

cliques $\to$ cliques

"complete subgraph"

$S =$ "cliques in G" is closed under:

G中选顶点集 $\tilde{T}$

$$\underline{G[\tilde{T}] \text{ complete}} \implies G[\forall T \subseteq \tilde{T}] \text{ complete}$$

由于 cliques 中每对顶点有且仅有一条连接，直觉

对于 $g: \underline{\mathrm{clique}(G)} \to \underline{\mathrm{clique}(H)}$，有唯一一对应关系

$$\{g_V(u), g_V(w)\} = g_S(uw) \in E_H$$

for $\forall w, u$ in V_G, $w \neq u$

于是，除去一般 simplical map 满足的 <u>identity</u> 与 <u>composition</u>，

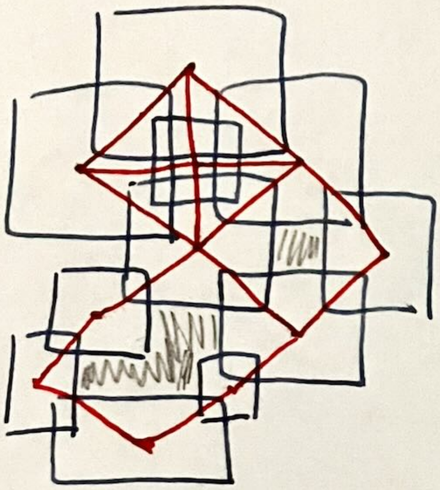
有：$A \cong B \implies \mathrm{clique}(A) \cong \mathrm{clique}(B)$

关于 Simplicial Complex 的应用：

假设有如下重叠 Blocks $\Rightarrow G = (B, \{b_ib_j \mid b_i \cap b_j \neq \emptyset\})$

b_i ： Blocks 的中心点代表每个 Block.

b_ib_j ： 若 Blocks overlaps，连为 Edges.



显而易见，coverage hole

只存在于 $|V| > 3$ 的 cycle 中间

定义 independence complex of G

$$ind(G) = (V_G,\ S = \{G[\sigma \subseteq V_G] \text{ has } \underline{\text{no Edges}}\})$$

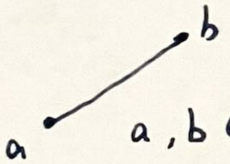
Hints :

$$ind(G) \cong clique(\overline{G})$$

顶点间无 edges

顶点间两两都有 edges.

回顾之前的 Geometric Realization 定义中，我们用线段表示 edges.



$a, b \in R^n$

$$\overline{ab} = \{(1-t)a + tb \mid t \in [0,1]\}$$

它可以改写为 **convex closure** 的形式

$$\text{conv}(ab) = \overline{ab} = \{t_0 a + t_1 b \mid t_0 + t_1 = 1,\ t_0, t_1 \geq 0\}$$

More generally,

$$\text{conv}(\sigma = \{v_0, v_1, \cdots\}) = \left\{\sum t_i v_i \,\middle|\, \sum t_i = 1,\ \forall_i\, t_i \geq 0\right\}$$

对于 Simplicial complex $K = (V_K, S_K)$

Each Vertices: $v_0 \dots v_{n-1}$

$$\mathrm{geom}(v_i) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \end{bmatrix}_i$$

Each Simplices:

$$\mathrm{geom}(\sigma) = \mathrm{conv}\left(\{\mathrm{geom}(v) \mid v \in \sigma\}\right)$$

where $\sigma \in S_K$

于是, $\mathrm{geom}(K) = \bigcup_{\sigma \in S_K} \mathrm{geom}(\sigma) \subseteq R^n$

Fact If $\sigma, \tau \in S_K$, then $\mathrm{geom}(\sigma) \cap \mathrm{geom}(\tau) = \mathrm{geom}(\sigma \cap \tau)$

即, "simplices intersect at a common subsimplex"

由于其在欧氏空间中的连续性，我们时常关心

Liner map $R^n \to R^d$ 的应用.

Let f be the restriction of $M \in R^{d \times n}$ to geom(K).

If f is <u>injective</u>, then we call this drawing an

linear <u>embedding</u> of K in R^d

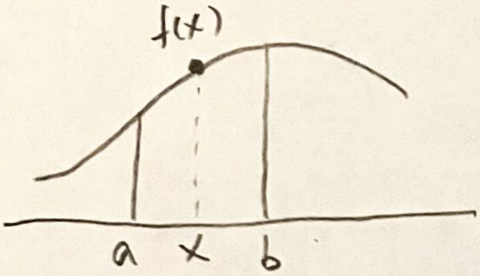
若存在这样的f，则 Simplical complex K 被称为

<u>geometric</u> simplical complex.

(i.e. 可具有拓扑结构)

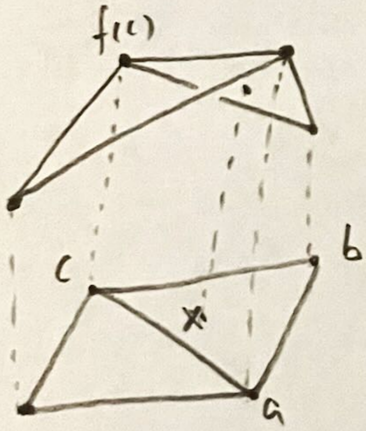
线性插值：通过2个已知数据点，构建直线

Interpolation 来计算中间未知点的值.



$$f(x) = \alpha f(a) + \beta f(b)$$

此方法可拓展到空间中：



$$f(x) = \alpha f(a) + \beta f(b) + \gamma f(c)$$

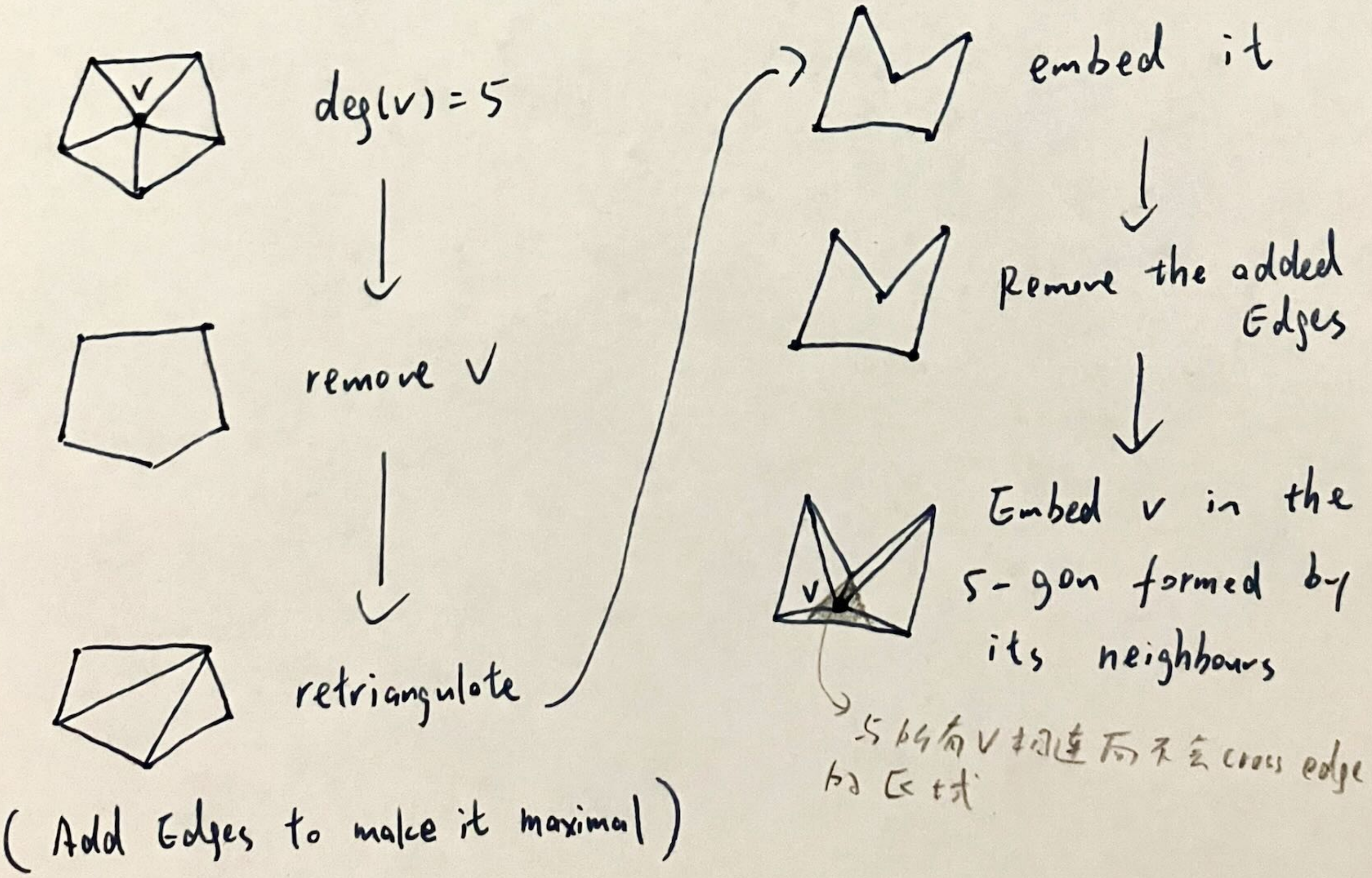
Fary's Theorem

If G is planar, then there exists a <u>linear embedding</u> $\text{geom}(G) \longrightarrow R^2$

"You can draw planar graphs with <u>straight lines</u>"

(加任意Edge将致Not planar)

前述，已知 <u>Maximal</u> planar graph 的每一种 Embedding 都是一种 Tringulation，依此从Euler's Formular 处推得 $|E| = 3|V| - 6$，可知任意 Vertex 的度 $\text{degree}(v) \le 5$.



对于其它 planar graph，可加入 Edges 使其 Maximal，依上述步骤 Embed 完毕再删掉加入的 Edges

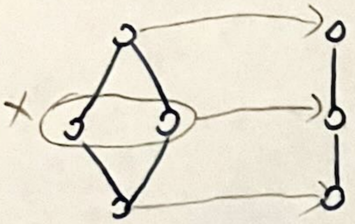
定义 A contraction is a surjective simplical map

$f: G \to H$ s.t. for all $H' \subseteq H$,

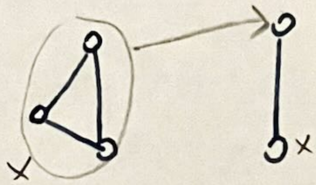
if H' is connected then $f^{-1}(H')$ is connected.

" Preimages of contracted subgraphs are connected "

反例: not contractions

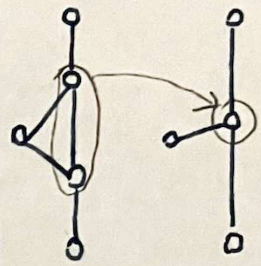


Disconnected Preimage



Not surjective

示例 Contraction



如何在 G 中 contract Edge uv ? $\boxed{G/uv}$ "Mod"

① remove vertex u and v

② Add a new vertex w

③ Links w with all old neighbors of u and v

写作 $G/uv = \begin{cases} V_G \setminus \{u, v\} \cup \{w\} \\ E_G \setminus \{e \mid u \in e \text{ or } v \in e\} \end{cases}$

$\cup \{ \text{"}xw\text{"} \mid x \text{ are old neighbors of } u/v \}$

Facts The composition of contractions is a contraction

证明:

有一组 contractions $A \xrightarrow{f} B \xrightarrow{g} C$

$g \circ f : A \to C$ is surjective because f, g are

Let $C' \subseteq C$ be a connected subgraph of C,

$g^{-1}(C')$ is connected (by definition of contractions)

$f^{-1}(g^{-1}(C'))$ is connected

于是 $(g \circ f)^{-1}(C')$ connected

2个条件满足，它是一个 contraction.

Fact If G is a subdivision of H, then there exist a contraction $G \to H$.

(recall : subdivision 指 break edges to paths)

之前所述 Topological Minors 也可视为通过 contract paths 操作得来，而广义 Minors 则可 contrast 任意 Edges

关于 Graph G "inside" Graph H 的一些情况

Subset

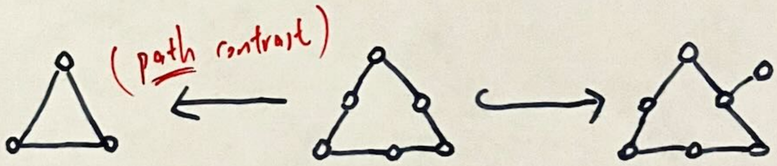
$G \subseteq H$

Subgraph

$G \hookrightarrow H$ "inclusion"

Topological Minor

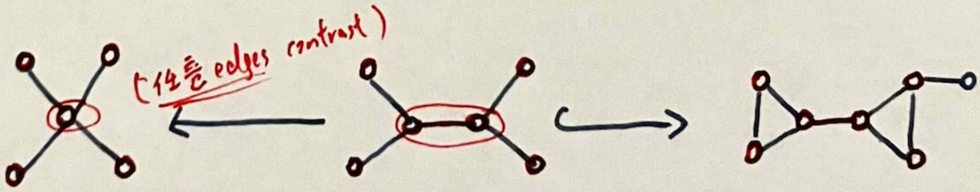
$G \preceq_T H$



$G \xleftarrow[\text{Constractions}]{\text{Topological}} H' \hookrightarrow H$

Minor

$G \preceq_M H$



$G \xleftarrow{\text{Contractions}} H' \hookrightarrow H$

很明显，Topological contractions 是一种特殊的 contraction.

$G \preceq_T H \implies G \preceq_M H$，但反之不成立

Minor ($\leq_m$) 作为 graph 上的 partial order 满足其中性质:

<u>Reflexive</u> : $G \leq_m G$ because id_G is a contraction

<u>Transitive</u> : $A \leq_m B \leq_m C \implies A \leq_m C$

$$
\begin{tikzcd}
 & & X \arrow[dl] \arrow[dr] & & \\
 & B' \arrow[dl] \arrow[dr] & & C' \arrow[dl] \arrow[dr] & \\
A \arrow[r, phantom, "\leq_m"] & & B \arrow[r, phantom, "\leq_m"] & & C
\end{tikzcd}
$$

Hints : $X \subseteq C'$, $B' \subseteq B$.

$\because$ $C' \to B$ 是 constraction

$\therefore$ $X \to B'$ is constraction

<u>Anti-Symmetric</u> : $A \leq_m B$ and $B \leq_m A$

imply $|V_A| = |V_B|$ and $|E_A| = |E_B|$

由于 $A \xleftarrow{con} B' \xhookrightarrow{includ} B$ 这个链条中, inclusion 与 contraction 按目前定义必须是 bijections, (不能改 V、E numbers 了)

于是 $A \cong B$

之前提过，当且仅当 不含 K_4 or $K_{2,3}$ subdivision 时，
(Topological minor)
G 是 outerplanar.

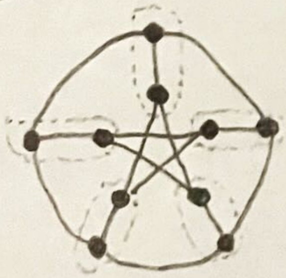
下述，当且仅当不含 K_5 or $K_{3,3}$ (Topological) Minor 时，
G 是 planar

Topological Minor　　　　　　　　　　Minor

$$\left[\begin{array}{c} K_5 \le_T G \\ \text{or} \\ K_{3,3} \le_T G \end{array}\right] \rightleftarrows \left[\begin{array}{c} K_5 \le_M G \\ \text{or} \\ K_{3,3} \le_M G \end{array}\right]$$

Kuratowski Therom $\Leftrightarrow$ G is not planar $\Leftrightarrow$ Wagner's Therom

示例



Peterson Graph P

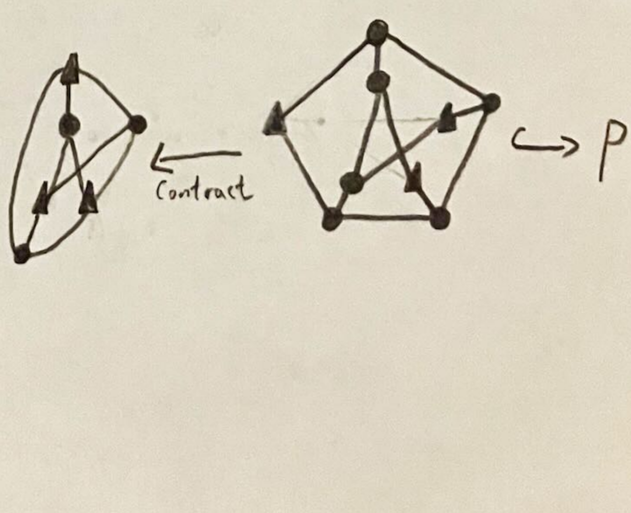
Not planar

$K_5 \not\le_T P$　∵ Max degree in P is 3, in K_5 is 4

$K_5 \le_M P$

$K_{3,3} \le_T P$

$K_{3,3} \le_M P$



证明 K_5 is not planar

∵ K_5 has 5 vertices and 10 edges

suppose K_5 is planar,

$$\begin{array}{ccc} |E| & & 3|V|-6 \\ 10 & \le & 3\times5-6 \end{array}$$

依 Euler's formular, $10 \le 3\times 5-6$ 不成立

证明 $K_{3,3}$ is not planar

子图 $K_{1,3}$ ("star S3") or $K_{2,2}$ ("cycle C4") connected

已知 $K_{3,3}$ 是 3-connected,

有 9 non-separating induced cycles,

one for each $K_{2,2} \cong C_4$ subgraph.

suppose $K_{3,3}$ is planar,

$$\begin{array}{ccccccccc} V & & E & & \text{faces} & & 1+K & & \\ 6 & - & 9 & + & 9 & = & 2 & & \end{array}$$

依 Euler's formular $6-9+9=2$ 不成立

定理 If H is nonplanar and $H \leq_T G$, than G is non planar

证明

$H \leftarrow G' \hookrightarrow G$

$H \leq_T G$ 意味着 $\exists G' \leq G$

$f : geom(H) \xrightarrow{\cong} geom(G')$

(而 H not planar)

假如 G planar,

就有 embedding $\phi : geom(G') \hookrightarrow R^2$

于是 $\phi \circ f : geom(H) \hookrightarrow R^2$ 也是 embedding

这样 H 就是 planar,

与假设相悖.

定义 Let $f: H \to G$ be a contraction.

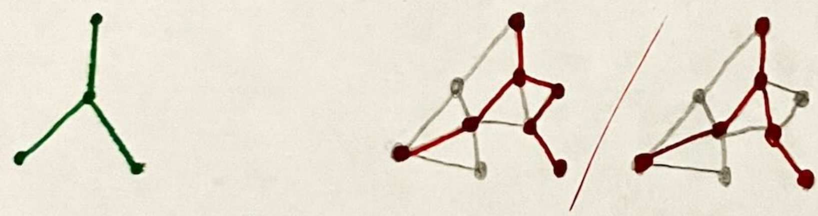
A <u>minimal preimage</u> of f is a minimal subgraph $S \subseteq H$ s.t. $f|_S$ is a contraction

(Minimal 含义：若preimage中移除再多，映射操作就不再是<u>contraction</u>)

不一定 unique!

Minimal preimage

$$G \xleftarrow{\text{Contractions}} \underline{\underline{S}} \hookrightarrow H$$



$G \leq_M H$

如上图，显然 Every $\underbrace{\text{tree with 3 leaves}}_{S}$ is a subdivision of $\underbrace{\text{Star(3)}}_{G}$ 于是：

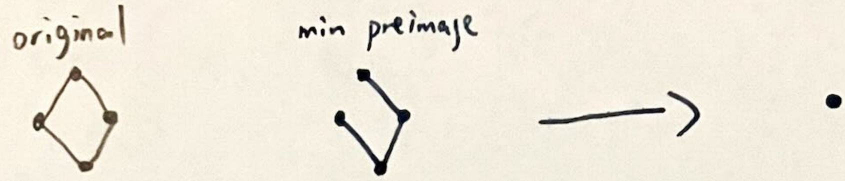
(Let G be a graph with maximum degree $= 3$,
If $G \leq_M H$ then $G \leq_T H$)

即，$\boxed{\text{Degree 3 minors are Topological}}$

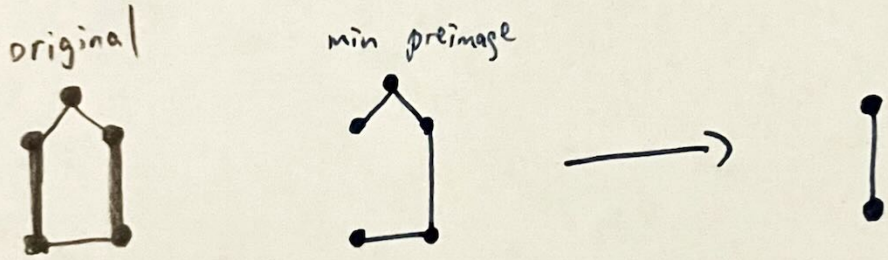
(仅 Topological contractions 即可得此 minor)

Facts about minimal preimages of contractions

The preimage of a vertex is a <u>tree</u>



The preimage of an edge is a <u>single edge</u>



If G is $\boxed{\text{connected}}$ and $|V| \geq 2$, then for all $e \in E$, G/e is connected.

证明:

Let $f: G \to G/e$ be the contraction

Take any $a, b \in G/e$

$a' \in f_V^{-1}(a)$

$b' \in f_V^{-1}(b)$ in original G

$\exists\ (a', b')$ walk $w: P_k \to G$

$f \circ w$ is an (a,b) walk in G/e

故, G/e is connected

$/$ contract

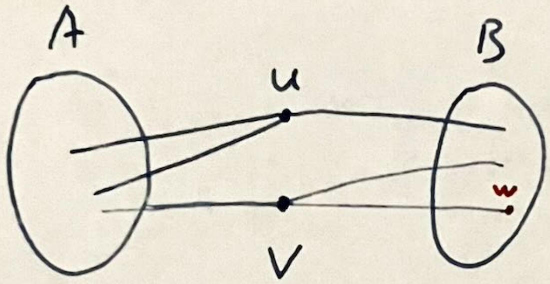
$\setminus$ remove

If G is $\boxed{\text{2-connected}}$ and $|V| \geq 4$, then there exist an edge e, G/e is 2-connected

证明:

假设 ~~存在~~ All $e=uv$ 使 G/e not 2-connected

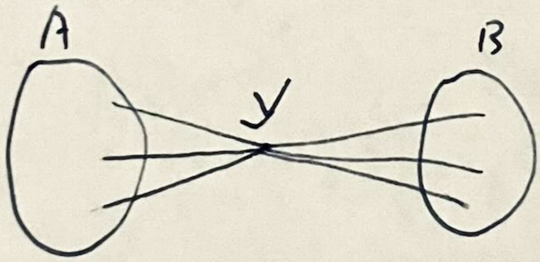
$\boxed{G}$



* u、v 选使得 $G \setminus \{u,v\}$ 中 largest component $\boxed{最大}$ 的组合. 这里指 A subgraph.

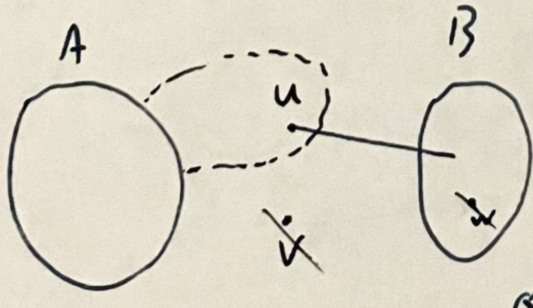
$G' \cong \boxed{G/uv}$

$(u \mapsto y$
$v \mapsto y)$



* not 2-connected 的唯一情形 (~~虽~~ still connected ∵ contraction)

Let w be a neighbour of v not in A.



$G \setminus \{v,w\}$ ($\mapsto y$) 依旧符合 not 2-connected 的唯一情形.

* 但! $|A \cup \{u\}| > |A|$

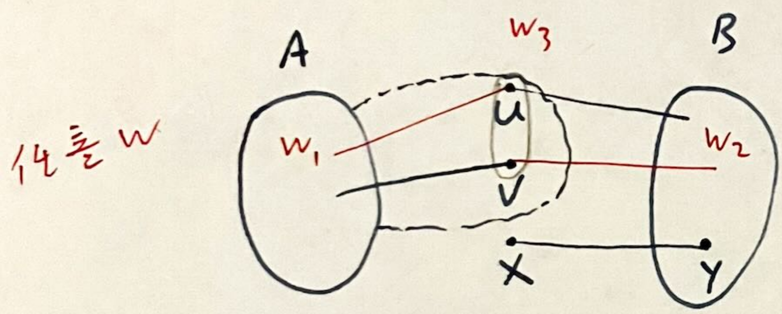
不是 $\boxed{\text{Maximal}}$ Largest component

于是假设不成立, G/e 必须是 2-connected

If G is 3-connected and $|V| \geq 5$, then there exist an edge e, G/e is 3-connected

证明.

假设 $e=uv$、点 x 可以使 $G \setminus \{u,v,x\}$ 中 largest component 最大的组合. 这里指 A subgraph

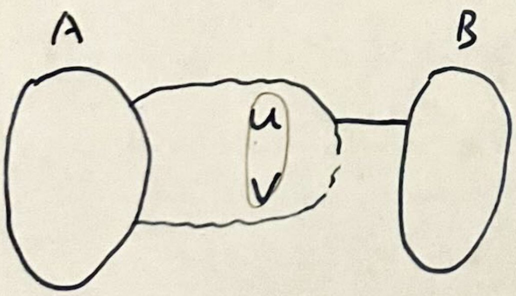


假设 ⇒ contract uv

G/e is not 3-connected (i.e. 2-connected)

⇒ $G/e \setminus \{x,y,w\}$

$G \setminus \{x,y,w\}$



(7-1)

connected ! not 2-connect

同样, $|A \cup \{u,v\}| > |A|$ 说明 A 不是

从设定的最大化 largest component

假设不成立,

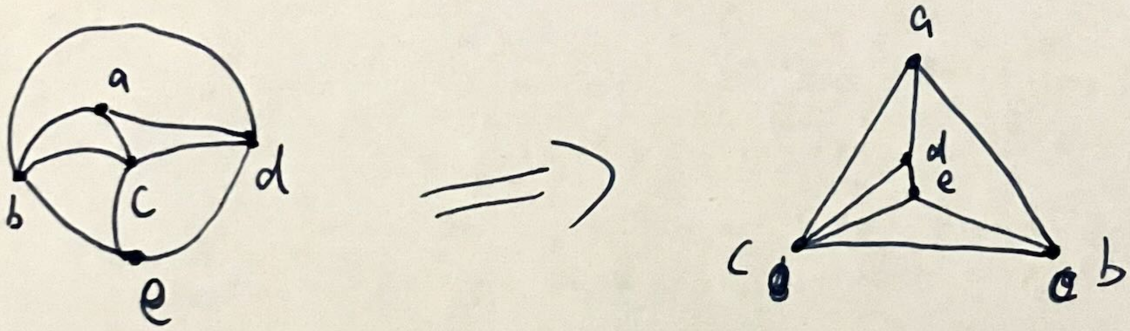
G/e 一定是 3-connected

同理, 归纳 k-connected 的情形

移动 degree(V)=5 的点，得到 Embedding

An extension to (Fary's Theorem)

Let G be a <u>maximal planar</u> graph, let abc be the vertices of some face. Then, there exists a <u>linear embedding</u> of G, s.t. Δabc is the <u>outer face</u>



Fary's Theorem 方法中，被移动的点只是在 face 中找位置。即，Δabc bounds a face，余下 V 在里找位点

关于 3-connected Graphs，它们至少有 4 vertices，正好可以形成 $\Delta abc + |V|$ 的这种应用 (?)

|V|=n 中，3个 vertex 作为 seperator，(n-3)个设为理论上最大 degree=5 再加 1

X 那一步思考，对于 3-connected graph，

不太对！

$$|E| = \frac{1}{2}\sum \text{Degree}(V)$$

$$\geq \frac{1}{2}\left[\underbrace{3}_{=} \times \underbrace{\boxed{3}}_{\boxed{2}+1} + \underbrace{(n-3)}_{=} \times \underbrace{\boxed{6}}_{\boxed{5}+1}\right] = 3n - 4.5 > 3n-6$$

not planar!

?!!!反了? —o— ✓

由 $|E|=3|V|-6$ 得

Recall: planar $\to |E| \leq 3|V|-6$

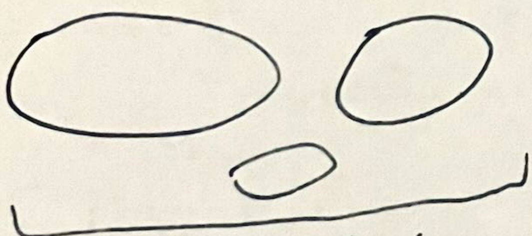
?

为何 Minimal Nonplanar Graphs are 3-connected?

Lemma

If every minor of G (other than G) is planar, then G is 3-connected

当 G is not connected 时

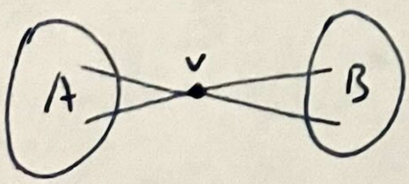


components of G

Components 各自 embedding, 即, G 可以是 planar

当 G connected with a cut vertex

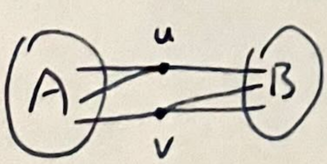
$\left[\begin{array}{l}\text{contract } B \longmapsto v \text{ 形成 } A' \leq_m G \\ \text{contract } A \longmapsto v \text{ 形成 } B' \leq_m G\end{array}\right]$



若二者皆 planar, 则 G 也 planar

"Embed A'、B' so that v is on the outerface"

当 G 2-connected with separator $\{u, v\}$



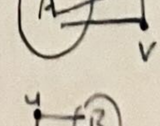
$\left[\begin{array}{l}\text{contract } B \longmapsto uv \text{ 形成 } A' \leq_m G \\ \text{contract } A \longmapsto uv \text{ 形成 } B' \leq_m G\end{array}\right]$

Both planar & Both have edge uv.

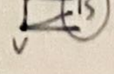
则 G 也 planar

(由 edge uv 卡住 2个 embedding)

A'

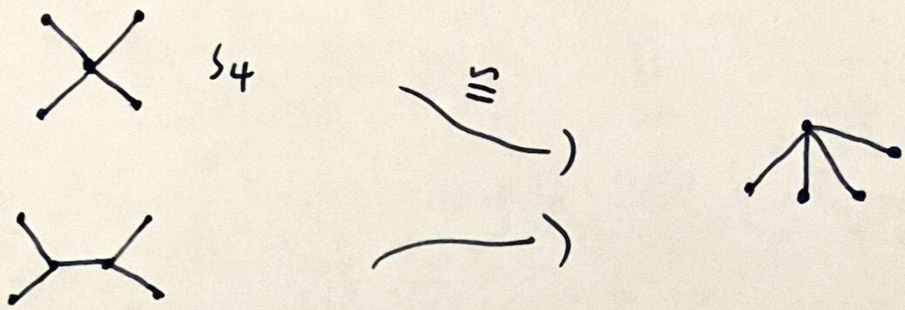


B'



Hints

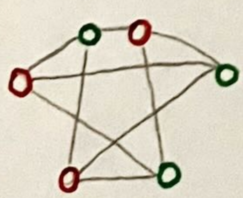
Trees with 4 leaves to minimal preimages {x 2 4}



$\boxed{\text{Case 1}}$ $K_5 \leq_T G$

All vertices homomorphic to S_4

$\boxed{\text{Case 2}}$ $K_{3,3} \leq_M G$



Lemma: If G is 3-connected and has no K_5 or $K_{3,3}$ minor, then G is planar.

Wagner's Therom

只是说 3-connected 可能 planar，也可能 not planar.

但前述，至少 3-connected 才能 not planar

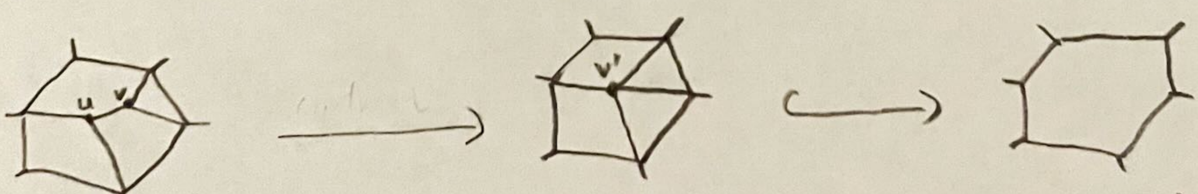
证明 Wagner's Therom：对于 3-connected graph G

从 base case ($|V|=5$) 开始归纳.

① $|V|=5$ 时，显然无 $K_{3,3}$ 子图

若要有 K_5 minor，其本身就是 K_5

② $|V|>5$ ($=6, 7, 8, \ldots$) 时，$\exists uv \in E$ s.t. G/uv is 3-connected



G $\longrightarrow$ G/uv $\longrightarrow$ $G \setminus v'$

3-connected $\Longleftarrow$ 2-connected

3-connected 无 $K_{3,3}$ or K_5 minor $\Longrightarrow$ 亦无 $K_{3,3}$ or K_5 Minors

（注：contraction 不会引入这两个 minors，除非 G 中本来就有）

总结一下.

若 G nonplanar

$(G' \xleftarrow{\text{contraction}} G)$

$\Rightarrow \exists$ minimal $G' \leq_M G$ that is non-planar

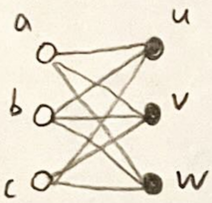
若 G' is 3-connected

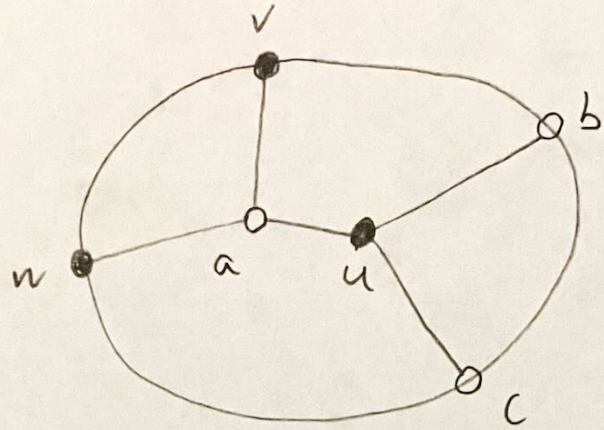
那只能是 $K_5 \leq_M G'$ or $K_{3,3} \leq_M G'$

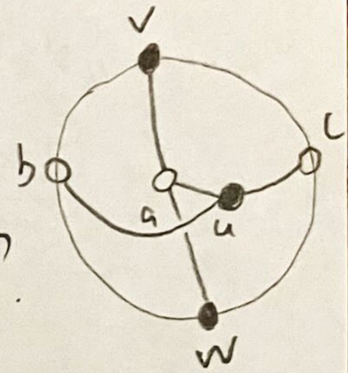
$\Rightarrow$ 即. G 中原本就有 K_5 or $K_{3,3}$ minor.

画一个外圈，圈上陈列 u、v 的 neighbors .

$\boxed{K_{3,3}}$

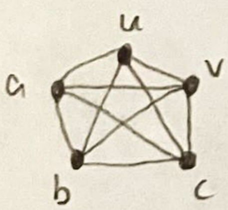


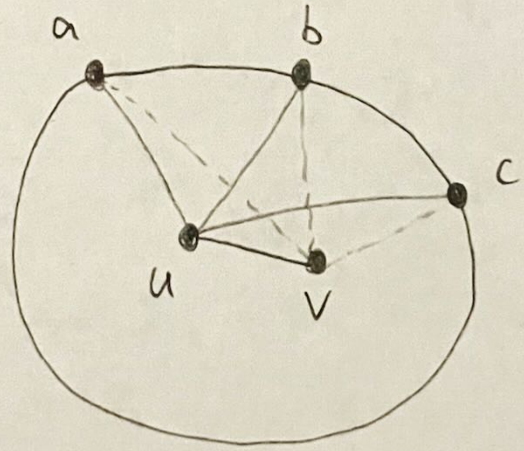




?? 视频中为何 ●○ 交替排列 ???

$\boxed{K_5}$





不用画图也知，

u、v 共同 neighbors 数目为 3 .

因此，若图中有<u>两点</u> u、v <u>共同 neighbors 数大于3</u>

<u>说明此图中有 K_5 minor</u>

满足如下变换，即是 $\boxed{\text{Linear Transformation}}$

$$f(x+y) = f(x) + f(y)$$

$$f(cx) = cf(x)$$

由上可知 $f(0)=0$，!!!

于是 $f(x) = mx + 3$ $\boxed{\text{not linear}}$

（任意 $b \neq 0$）

例）$id_V : V \to V$ is linear

证 $id_V(u+w) = u+w = id_V(u) + id_V(w)$

$id_V(cu) = cu = c\, id_V(u)$

例）. The composition of linear maps is linear

$$U \xrightarrow{g} V \xrightarrow{f} W$$

证 $(f\circ g)(x+y) = \cdots\cdots = (f\circ g)x + (f\circ g)y$

$(f\circ g)(cu) = \cdots = c\,(f\circ g)(x)$

Let F be a field，于其中 + − * / 都 "work"

e.g. $\mathbb{R}$、$\mathbb{C}$、$\mathbb{Q}$、$GF(2)$

A $\boxed{\text{vector space } V}$ over F is a set with the following 2 operations for $\left(\begin{matrix} u, v, w \in V \\ a, b \in F \end{matrix}\right)$

Addition $u + v$

$$(u+v)+w = u+(v+w)$$
$$u+v = v+u$$
$$\exists 0 \text{ vector s.t. } v + \overset{\text{"zero"}}{0} = v$$
$$\exists (-v) \text{ s.t. } v+(-v) = 0$$

scalar multiple av

$$a(bv) = (ab)v$$
$$1v = v$$
$$a(u+v) = au + av$$
$$(a+b)v = av + bv$$

(13.) $v \in \mathbb{R}^n$, $a \in \mathbb{R}$

(13.) For a finite set S,

$$(Pow(S), \oplus)$$

($Pow(S)$ forms a vector space with symmetric difference
singletons $\{x\}$ for $x \in S$ form a $\boxed{\text{basis}}$)

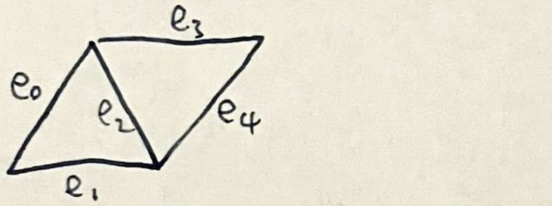
The edge space of graph G is

$$\mathcal{E}_G = (\underbrace{Pow(E)}_{\text{subsets of Edges}}, \underbrace{\oplus}_{\substack{\text{symmetric difference} \\ \text{e.g. addition}}})$$

$\dim(\mathcal{E}_G) = |E|$

which is a $\boxed{\text{vector space}}$ over $\underline{GF(2)}$

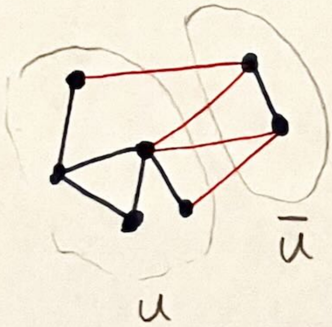
$$1 + 1 = 0$$
$$1 = -1$$
$$a + b = a - b$$



$$\{e_0, e_3\} \oplus \{e_2, e_3\} = \{e_0, e_2\}$$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \oplus \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

(Edge) Cuts in a Graph



一组顶点

For subset $u \subseteq V_G$,

cross product 对 dim>3 不唯一! 先忽略此行

$$cut(u) = E_G \cancel{\cap (u \times \bar{u})}$$

$$= \{ e \in E_G \mid |e \cap u| = 1 \}$$

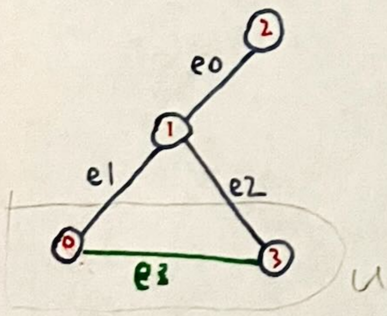
[Eedg 仅有一个端点在 u 中]

"从G中切下u所需断的Edges"

$$cut : V_G \longrightarrow \mathcal{E}_G^{\boxed{GF(2)}}$$

是一个Linear变换

$cut(A \oplus B) = cut(A) \oplus cut(B)$



$$B = \begin{array}{cc} \begin{array}{ccc} e0 & e1 & e2 \end{array} & e3 \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{array}{c} 1 \\ 0 \\ 0 \\ 1 \end{array} \end{array}$$

$b_{ij} = 1$ if $v_i \in e_j$

$= 0$ otherwise

$$cut(u) = B^T u$$

$$= \begin{array}{c} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \\ \begin{array}{cccc} 1 & 0 & 0 & 1 \end{array} \end{array} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \; 2 \Downarrow 0$$

若edge 2两端都在u中，

会被 1+1=0 cancel 掉

$$B : \mathcal{E}_G \longrightarrow \mathcal{V}_G$$

$$B^T : \mathcal{V}_G \longrightarrow \mathcal{E}_G$$

The cut space of G

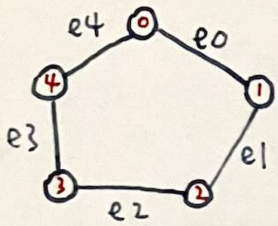
$$C^*_G = \operatorname{im} B^T$$

is a subspace of $\mathcal{E}_G$

If x is the set of edges in a cycle,

then $Bx = 0$ Each edge counted twice $\rightarrow 1+1=0$
(2个端点.)

$\boxed{\text{Cycle space } C_G} = \ker B = \{ x \in \mathcal{E}_G \mid Bx = 0 \}$



$$
\begin{array}{c}
\begin{array}{ccccc} e0 & e1 & e2 & e3 & e4 \end{array} \\
Bx = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} \overset{GF(2)}{\Longrightarrow} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\end{array}
$$

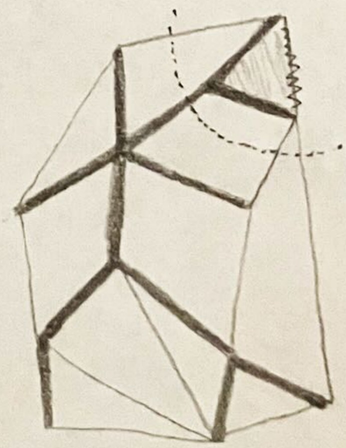
~~$x \in C_G$ iff~~ cycles $\Longleftrightarrow$ even degree graphs

Cycle space C_G is generated by edge sets of cycles

Recall: Basis(基)是一组线性无关的元素，生成整个空间

如何将一个Graph拆成许多小cycles ("A basis of cycles")

1. Spanning Tree of G
2. 加入一条边
   生成一个cycle
3. 切掉这个cycle Loop



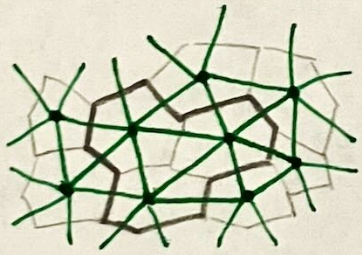
证明见 $\boxed{\text{上一页}}$

$$\boxed{C_G \cong C^*_{G'}}$$

Cycle space of G is isomorphic to the cut space of its dual G'

Recall planar 3-connected graph 才有 dual

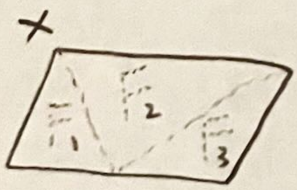
Recall Original (Map) ⟷ Dual (G')
"cycles" "cuts"



Cycle 是一种 Jordan Curve
沿 Space 切为内/外

Let G be a 2-connected planar graph embedded in the plane. The bounded-face cycles of G form a basis of C_G

证 设 X 是某个 Simple cycle 的 Edges set



$$X = F_1 \oplus F_2 \oplus F_3$$

$\boxed{GF(2)}$

(Counts for each edge : 1+1=0.)

More General

Let G be planar, Let $\underline{G}$ be its embedding,

Let $\underline{G}'$ be the dual of the embedding.

$$\boxed{C_G \cong C^*_{\underline{G}'}}$$

$\underline{G}'$ is an abstract dual of G

existed iff G is planar

证

对于任意两个 finite dimensional 的 vector space，若

二者具有相同的维数，则它们 isomorphic $\cong$

已知

Cycle Space C_G	Cut Space C^*_G
$\mathrm{Dim} = \lvert E\rvert - \underbrace{(\lvert V\rvert - k)}_{\text{\#edges in the spanning tree}}$	$\mathrm{Dim} = \lvert V\rvert - k$ → # components

$$\mathrm{Dim}(C_G) + \mathrm{Dim}(C^*_G) = \mathrm{Dim}(\mathcal{E}_G)$$

Original (G) ⟶ Dual (G')

"Faces" ⟶ "Vertices"

$$\mathrm{Dim}(C_G) = |E_G| - (|V_G| - 1)$$

"Euler's Formular"

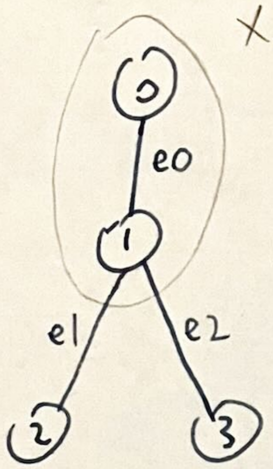
$$= |F_G| - 1$$

$$= |V_{G'}| - 1 = \mathrm{Dim}(C^*_{G'})$$

$$= |V_{\underline{G}'}| - 1 = \mathrm{Dim}(C^*_{\underline{G}'})$$

故 Isomorphic

Matrices from Graphs.



$$X = \begin{array}{c} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \end{array} \begin{array}{l} v_0 \\ v_1 \\ \\ \\ \end{array}$$

$$Y = \begin{array}{c} e_0 \\ \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \end{array}$$

$$\begin{array}{c c} & \begin{array}{ccc} e_0 & e_1 & e_2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \end{array}$$

GF(2)中，也可把-1写作1，因为1+1=0

Boundary

(前述B)

$\partial : \mathcal{E}_G \to \mathcal{V}_G \quad \partial y$ for $y \in \mathcal{E}$

$\partial^T : \mathcal{V}_G \to \mathcal{E}_G \quad \partial^T x$ for $x \in \mathcal{V}$

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \quad \text{Laplacian} = \partial\partial^T : \mathcal{V}_G \to \mathcal{V}_G$$

$$= \begin{bmatrix} 1 & & & \\ & 3 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$= \text{degrees} - \text{adjacency}$$

$$[\partial\partial^T]_{ik} = \sum_{j \in |G|} \partial_{ij}\, \partial^T_{jk}$$

$$= \begin{cases} \deg(v_i) & \text{if } i = k \\ -1 & \text{if } v_i v_k \in E \\ 0 & \text{otherwise} \end{cases}$$

Ohm's Law : $w = \frac{v}{r}$

w - current
v - voltage
r - resistance
p_a - potential at V_a
c - net current

if Assume $r=1$

$w = v$ △1

a •——• b

$$\left(\begin{aligned} w \in R^{|E|} &\rightsquigarrow c = \partial w \in R^{|V|} \\ p \in R^{|V|} &\rightsquigarrow v = \partial^T p \in R^{|E|} \end{aligned}\right) \quad △2$$

由 △1 与 △2 得 $\partial^T p = w$

"Laplacian Mtx"

于是 $c = \partial w = \partial\partial^T p = Lp$

又知 $(Lx)^T \mathbf{1} = \sum_i \sum_j (x_i - x_j) = 0$ $\quad (i,j \in E)$

[illegible] 12.1

故，当 $c^T \mathbf{1} = 0$ 时，$Lp = c$

又，如何 $\boxed{求解\ p}$？

固定 $c = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$ s : 电流 In；t : 电流 Out

~~$Lp = p_s - p_t$~~

~~为函数~~

找一个固定的 $Lp = c$

$L^+ L p = L^+ c$

$L(p + \mathbf{1}) = Lp + L\mathbf{1}$
$= Lp$

Permutation Matrix P 交换 vector 元素的顺序

$$Px = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} d \\ b \\ a \\ c \end{bmatrix}$$

P_{ij} : 将 V_i 挪到 V_j 位置.

其 Inverse $P_{ij}^{-1} = P_{ji}$ 即 $P^{-1} = P^T$

故，Permutations are Isomorphisms $\cong$

$$B = \partial = \begin{array}{cc} & |E| \\ |V| & \begin{bmatrix} e_j & \\ \vdots & - v_i - \end{bmatrix} \end{array}$$

$P_{|V|}\partial$ rearrange vertices

$\partial P_{|E|}$ rearrange edges

Cycle / cut space is invariant to vertex/edge ordering

$$C_G = \ker \partial \cong \ker P\partial .$$

$$C_G^* = \mathrm{im}\, \partial^T \cong \mathrm{im}\, P_{|E|} \partial^T P_{|V|}$$

对于 G 及其 Dual G'

$$
\begin{tikzcd}
V \arrow[r, shift left, "\partial_G^T"'] & E \arrow[l, shift left, "\partial_G"'] \arrow[r, shift left, "\partial_{G'}"] & F \arrow[l, shift left, "\partial^T_{G'}"]
\end{tikzcd}
$$

Cut Space　　　　　　　　　　　　　　Cycle Space

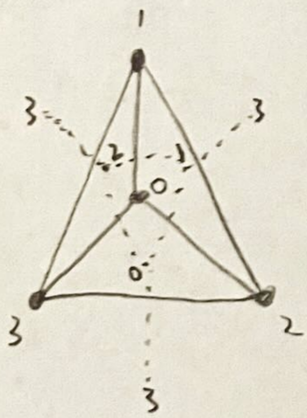
$$C^*_G = \operatorname{im}\partial_G^T \underbrace{\cong} \ker \partial_{G'} = C_{G'}$$

边的顺序、方向不同，其余相同

如何为 Dual G' 分配方向？

（为 $\boxed{\partial_G}$ 分配 ± 1）signs

$$\partial_{G'}\partial_G^T \overset{?}{=} 0$$



$$
\partial_{G'} \quad \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}
\qquad
\partial_G^T \quad \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}
$$

Sub - Laplacian

若将G分为2个子图，Laplacian Matrix 的用法：

$$\left[\begin{array}{c|c} L_0 & Q^T \\ \hline Q & L_1 \end{array}\right] \left[\begin{array}{c} P_0 \rightarrow \text{Fixed} \\ \hline P_1 \rightarrow ? \end{array}\right] = \left[\begin{array}{c} \text{Don't care} \\ \hline 0 \end{array}\right]$$

$$L \left[\begin{array}{c} \text{Fix} \\ \hline ? \end{array}\right] = \left[\begin{array}{c} \backslash \\ \hline 0 \end{array}\right]$$

记之，

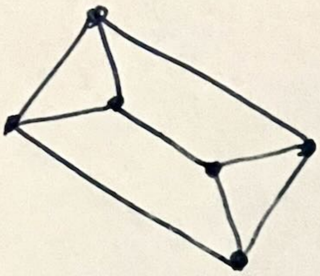
$$Q P_0 + L_1 P_1 = 0$$

$$\underline{P_1 = -L_1^{-1} Q P_0}$$

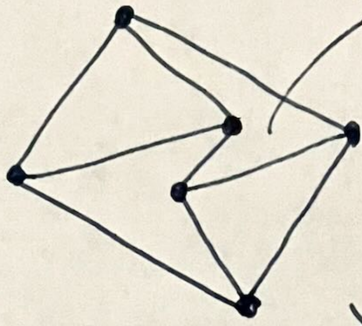
Recall: $L = [\text{Degree}] - [\text{Adj}]$

$$L_1 = \underbrace{L_S}_{\text{Laplacian of subgraph } S} + \underbrace{X}_{[\text{Degree}] \text{ of } E_G \setminus E_H}$$

定义 A convex embedding of a planar 3-connected graph is one in which every face is a convex embedding



✓



×

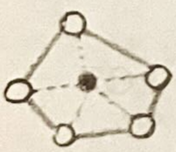
Monotone paths

"当力不平衡时，沿 Edges 走，直到平衡"
至多停于 Outerface
& 力来自 Outerface

证 13.5 videos 略.

Tutte's Algorithm 为画 convex embedding 的方法.

思想：固定一个 outer face，余下 Edges 都是弹簧



Force on $\overset{i}{\bullet}$ is $\sum_{\circ} (\bullet - \circ)$ with all its neighbor $\overset{j}{\circ}$

Let $P \in R^{n \times 2}$ (x, y) be the positions of the n vertices in the plane.

$$[LP]_i = -\sum_{j} (P_j - P_i)$$

$\cancel{\underbrace{\partial S}_{\text{弹簧系数}} \partial^T P = \underbrace{0}_{\text{Not force}}}$ 见下页

$$L \begin{bmatrix} P_{fix} \\ P_{inner} \end{bmatrix} = \begin{bmatrix} \sim \\ 0 \end{bmatrix} \Longrightarrow L_1 P_{inner} = -Q P_{fix}$$

公式，直接用

$$\boxed{P_{inner} = -L_1^{-1} Q P_{fix}}$$

用 ds2viz 包时，

from —— . canvas import Canvas
from —— . datastructures import VizGraph

G = generate_polytopal_graph (120)

P = [: ,[1, 2]] set of positions

e.g. from eig_vects

e.g. from $-L_1^{-1} Q P_{fix}$

c = Canvas (500, 500)

VizGraph (G , P). draw (c)

print (c. svgout())

GTAC 13

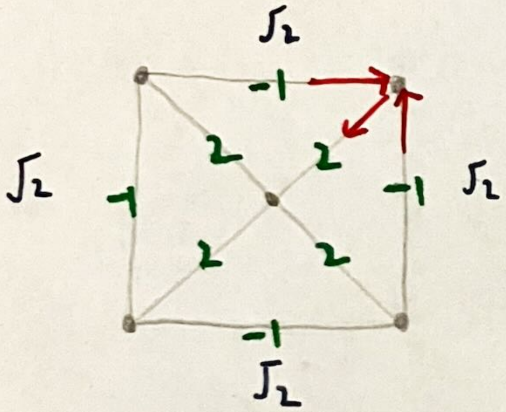
另一种理解Tutte's的方式：$\boxed{\text{Equilibrum stress}}$

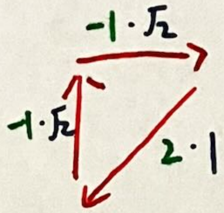
Treat edge e as a spring with spring constant S_e

$$\partial \underline{S} \partial^T \underline{p} = 0 \text{ / Net Force}$$

positions of points

$L = \partial\partial^T \quad \begin{bmatrix} S_0 & & \\ & \ddots & \\ & & S_{|E|-1} \end{bmatrix}$





$\Sigma S_e \cdot \text{length}_e$

$= 0$

G : planar , 3-connected , positions $P \in R^{|V| \times 2}$

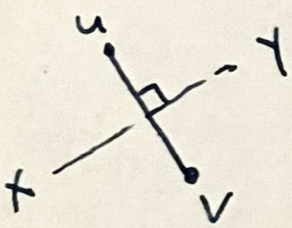
H : dual of G , positions $f \in R^{|F_G| \times 2}$

($\text{Faces}_G \rightarrow \text{Vertices}_H$)

are $\boxed{\text{reciprocal diagrams}}$ if

性质

$\forall$ edges $\langle \overset{\text{in } G}{uv}, \overset{\text{in } H}{xy} \rangle$ 相互垂直



$$(P_u - P_v)^T (f_x - f_y) = 0$$

此时，存在一个 scalling s_e，使得 Rotation 90°(R) 后，$R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$f_x - f_y = s_e (p_u - p_v) R$$

归纳得 $d_H^T f = S d_G^T p R$

$$d_G S d_G^T p = \underbrace{d_G d_H^T}_{L=0} f R^{-1} = 0$$

$\boxed{\text{Equilibrum stress}}$

证①

Equilibrum Stress $\longrightarrow$ Reciprocal Diagram.

Suppose $\partial_G S \partial_G^T p = 0$

want f s.t. $\partial_H^T f = \underline{S \partial_G^T p R}$

已知 $\operatorname{im} \partial_H^T = \ker \partial_G$

即，若 f 存在，$\underline{S\partial_G^T pR}$ 应该在 $\ker \partial_G$ 中，

即 $\underbrace{\partial_G \cdot S \partial_G^T}_{0} pR = 0$

成立！

证②

The Maxwell - Cremona Correspondence

Equilibrum Stress

(Spring constants)

Liftings

(height function)

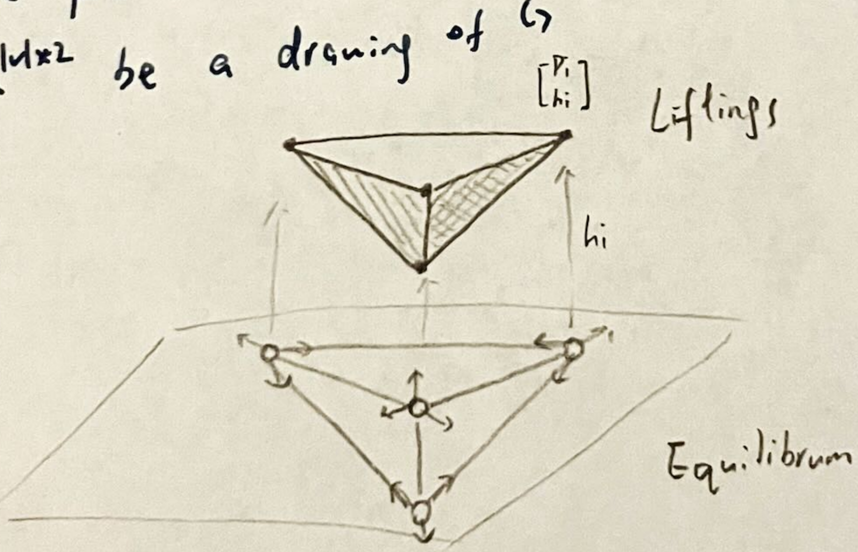
证① 证② 证③ 证④

Reciprocal Diagram

(Dual Drawing)

Let G be planar, 3-connected

Let $P \in R^{|V| \times 2}$ be a drawing of G



定义 A $\boxed{\text{Lifting}}$ of G is $h \in R^n$ s.t.

for all faces x, for all vertices v_i in x,

the points $\begin{bmatrix} p_i \\ h_i \end{bmatrix} \in R^3$ lie on a plane 位于同一平面

R^3中 All faces stay $\boxed{\text{flat}}$

$f_x((p_1-p_2)+(p_2-p_3)+(p_3-p_1))=0$

$h \to f$

$\boxed{\text{Lifting} \to \text{Reciprocal Diagram}}$ 证③

For each face x, the lifted plane has gradient f_x

(plane 的倾斜度?)

Gradient, vector, height variance

$$f_x^T(p_u - p_v) = h_u - h_v = f_y^T(p_u - p_v)$$

即 $(f_x - f_y)^T(p_u - p_v) = 0$ 正交的

可由 Lifting 得到 Reciprocal Diagram

$f \to h$

$\boxed{\text{Reciprocal Diagram} \to \text{Lifting}}$ 证④

Given f, 求 $\forall$ edge $\langle uv, xy \rangle$ 中 h 的 changes

$$b_e = h_u - h_v = f_x^T(p_u - p_v)$$

即 $b = \partial_G^T h$ 已知 $\operatorname{im}\partial_G^T = \operatorname{Ker}\partial_H$

即 $[\partial_H b]_x = \sum_{\substack{\langle uv, xy\rangle \\ \text{for } x}} f_x^T(p_u - p_v) = f_x^T \underbrace{\partial_H \partial_G^T}_{0} p = 0$

成立

b 确实在 $\operatorname{Ker}\partial_H$ 中 (b_e 有解)

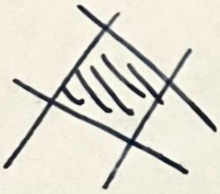
$\boxed{\text{Steinitz's Theorem}}$

G or dual G'，至少有一个含△三角形!

Every $\boxed{\text{planar, 3-connected } G}$ can be realized as the edges of a convex polytope in 3D.

$\boxed{\text{Polyhedra}}$ are intersections of halfspaces

e.g. $\{x \mid n^T x \le 3\}$ 



$\boxed{\text{Polytopes}}$ are bounded polyhedra

如何 Lifting ? (planar → 3D polytopes)

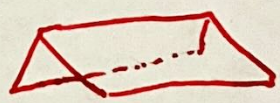
MCC

3D Tutte ?

Valley

(positive stress)

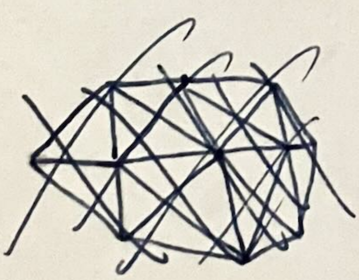
Mountain

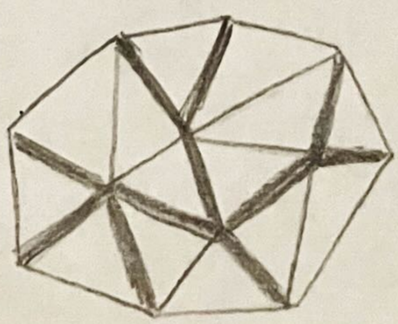


(Negative stress)

复杂图形 : 利用 Spanning Tree

There is a spanning tree that doesn't contain edges of the other face





△三角形不论怎样 Lifting，都一定 flat.